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A TOPOLOGICAL PROOF OF THE RIEMANN-ROCH THEOREM ON AN ALGEBRAIC CURVE.

By Oscar Zariski.

The purpose of this paper is to develop a *topological theory of linear series on an algebraic curve.* Our chief tool is the *symmetric topological product* of the Riemann surface of the curve: its use was clearly indicated by the involutorial character of linear series and plays here the same rôle as the notion of the direct topological product plays in Lefschetz's topological theory of algebraic correspondences (see S. Lefschetz (3); also Todd (5), Zariski (6), chapter VI and Appendix B). It will be seen from the following exposition that the use of symmetric products in the geometry on an algebraic curve leads very rapidly to the very heart of the theory, including the central theorem of Riemann-Roch.

1. By the *symmetric n-th product* of a complex K is meant the topological space of the *unordered* sets of n points of K. This space, which we shall denote by K^n, is in $(1, n!)$ correspondence with the direct topological product of n complexes homeomorphic to K. The involution of sets of $n!$ points defined by this correspondence on the direct product is generated by a group G of automorphisms T_ν, isomorphic with the symmetric group of substitutions on n letters. From this point of view symmetric products were studied recently by M. Richardson,[4] who gave a convenient simplicial subdivision of K^n, outlined a general procedure for the determination of its Betti numbers and determined explicitly these numbers in the cases $n = 2$ and $n = 3$.

In this paper we shall be concerned with the symmetric n-th product R^n of a Riemann surface R of genus p. If $p = 0$, R is a 2-sphere and R^n is the space of all unordered sets of n (distinct or coincident) complex numbers $z_1, z_2, \cdots, z_n$, including $z = \infty$. With any such set we associate the coefficients, determined to within a factor of proportionality, of the polynomial $a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ of which the numbers of the set are the roots. If k of the numbers of the set coincide with $z = \infty$, then $a_0 = a_1 = \cdots = a_{k-1} = 0$. It follows that *in the case $p = 0$ R^n is homeomorphic with the complex projective n-dimensional space.* Since the manifold condition is of a local character, it follows that R^n *is an absolute manifold* also if $p > 0$.

We make a few preliminary remarks concerning the boundary and orientation of the symmetric product of a 1-cell and of a 2-cell.

R.J.

Let the 1-cell E_1 be the segment $(0 - 1)$ of the real line x. It is seen immediately that E_1^n is homeomorphic with the following point set in the Euclidean space $(x_1, x_2, \cdots, x_n)$: $0 < x_1 \leqq x_2 \leqq \cdots \leqq x_n < 1$. This point set is a simplex σ_n whose $n + 1$ faces lie on the hyperplanes $x_1 = 0$, $x_n = 1$, $x_i = x_{i+1}$, $(i = 1, 2, \cdots, n - 1)$. Of these faces only the first two arise from the boundary of E_1, while the remaining faces arise from the n-tuples of points of E_1 in which two or more points coincide. In this circumstance lies the reason of the fact that the n-th symmetric product of a 1-sphere is only a relative manifold (a manifold with a boundary). Thus, for $n = 2$, it is the Moebius ring.

On the contrary, the n-th symmetric product E_2^n of a 2-cell E_2 is a $2n$-cell whose boundary arises exclusively from the boundary of E_2, its points corresponding to those n-tuples of points of the closed E_2 which have at least one point on the boundary of E_2. To see this it is sufficient to introduce in E_2 a complex parameter z and to apply the method of symmetric functions of the roots $z_1, z_2, \cdots, z_n$ used above. For this reason the symmetric product of an absolute manifold M_2 is an absolute manifold.

For a given orientation of E_2 a corresponding orientation of E_2^n may be defined as follows. Let $(P_1, P_2, \cdots, P_n)$ be a set of n distinct points of E_2 and let (ξ_i, η_i) be an indicatrix of the oriented E_2 at P_i. The symbol $(\xi_1, \eta_1; \xi_2, \eta_2; \cdots; \xi_n, \eta_n)$ defines in an obvious manner an indicatrix at the point $P \equiv (P_1, P_2, \cdots, P_n)$ on E_2^n and hence an orientation of E_2^n. *This orientation is independent of the order of the points P_i of the n-tuple,* because any permutation of the points P_i induces an even permutation of the elements of the indicatrix at P. As a corollary it follows that *R^n is an orientable manifold.*

An analogous consideration for the orientation of E_1^n leads to an indicatrix $(\xi_1, \xi_2, \cdots, \xi_n)$ which is altered by an odd permutation of the points P_i. Since on a 1-sphere H_1 it is possible to deform into each other any two *ordered* n-tuples of distinct points of H_1 without crossing n-tuples having coincident points, it follows that the relative manifold H_1^n is non orientable.

2. *Cycles and minimal bases on R^n.* It has a sense to speak of the symmetric product also when the factors $K_1, K_2, \cdots$ are distinct subcomplexes of a given complex K or chains on K. When these subcomplexes or chains have cells in common their symmetric product is in general different from the direct topological product. While keeping the usual notation $K_1 \times K_2 \times \cdots$ for the direct topological product, we shall denote the symmetric product by $K_1 K_2 \cdots$.

Let $M_{2n} = R \times R \times \cdots \times R$ be the direct topological product of n

factors R and let us consider the correspondence $(1, n!)$ between R^n and M_{2n}. We denote by $\phi(P')$ the homologue P on R^n of a point P' on M_{2n} and by $\psi(P)$ the set of $n!$ points on M_{2n} which correspond to a given point P on R^n. The operations ϕ and ψ transform chains into chains and preserve boundary relations. In particular, they transform cycles into cycles and homologous cycles into homologous cycles (see M. Richardson (4)). If C is a chain on R^n, then $\phi\psi(C) = n!C$.

Let $\Gamma_k^1, \Gamma_k^2, \cdots$ be a maximal set of independent k-cycles on M_{2n} and let $\Delta_k^i = \phi(\Gamma_k^i)$ be the corresponding cycles on R^n. Let Δ_k be an arbitrary k-cycle on R^n and let $\Gamma_k = \psi(\Delta_k)$ be the corresponding cycle on M_{2n}. We have $t\Gamma_k \sim t_1\Gamma_k^1 + t_2\Gamma_k^2 + \cdots$. Operating by ϕ and observing that $\phi(\Gamma_k) = n!\Delta_k$, we obtain: $n!t\Delta_k \sim t_1\Delta_k^1 + t_2\Delta_k^2 + \cdots$. Hence *every k-cycle on R^n depends on the cycles Δ_k^i.*[1]

Let $\delta_1, \delta_2, \cdots, \delta_{2p}$ be a minimal base of 1-cycles for homology $\sim$ on R. It is well known that the following k-cycles on M_{2n} form a minimal base for homology $\sim$:

$$\Gamma_k^{(i)} = \Gamma_k^{i_1, i_2, \cdots, i_\beta} = M_{2\alpha} \times \delta_{i_1} \times \delta_{i_2} \times \cdots \times \delta_{i_\beta} \times P_1 \times P_2 \times \cdots \times P_\gamma,$$
$$2\alpha + \beta = k, \qquad \alpha + \beta + \gamma = n,$$

where $M_{2\alpha}$ is the direct product of α factors R and $P_1, P_2, \cdots, P_\gamma$ are arbitrary fixed points on R. Of what nature are the corresponding cycles $\phi(\Gamma_k^{(i)})$ on R^n? Let us first consider the case in which the indices $i_1, i_2, \cdots, i_\beta$ are all distinct. In this case the correspondence between $\phi(\Gamma_k^{(i)})$ and $\Gamma_k^{(i)}$ is $(1, \alpha!)$ and the cycle $\Gamma_k^{(i)}$ is transformed into itself by the $\alpha!$ automorphisms T_ν of M_{2n} (see section 1) which correspond to the permutations of the α factors R in $M_{2\alpha}$. *These automorphisms preserve the orientation of* $\Gamma_k^{(i)}$, since the permuted factors are of even dimension. It follows that in $\phi(\Gamma_k^{(i)})$ each oriented cell is repeated $\alpha!$ times and that consequently,

$$\phi(\Gamma_k^{i_1, i_2, \cdots, i_\beta}) = \alpha!\,\Delta_k^{i_1, i_2, \cdots, i_\beta} = \alpha!\,\Delta_k^{(i)},$$

where $\Delta_k^{(i)}$ is a cycle on R^n which as a locus of points consists of all the unordered n-tuples of points of R having one point on δ_{i_1}, one point on $\delta_{i_2}, \cdots$, one point on δ_{i_β} and γ points coincident with $P_1, P_2, \cdots, P_\gamma$. In our notation for symmetric products we have

(1) $$\Delta_k^{(i)} = \Delta_k^{i_1, i_2, \cdots, i_\beta} = R^\alpha \delta_{i_1}\delta_{i_2} \cdots \delta_{i_\beta} P_1 P_2 \cdots P_\gamma,$$

where

(2) $$2\alpha + \beta = k, \qquad \alpha + \beta + \gamma = n.$$

[1] In M. Richardson (4) the above proof is developed along strictly combinatorial lines.

The orientation of $\Delta_k^{(i)}$ is determined by the orientation of $\Gamma_k^{(i)}$ and depends on the orientation of R and on the orientation and on the order of the cycles $\delta_{i_1}, \delta_{i_2}, \cdots, \delta_{i_\beta}$.

Let now $\Gamma_k^{(i)}$ be a cycle with repeated indices, for instance, let $i_1 = i_2$. The involutorial automorphism T of M_{2n} which interchanges the factors δ_{i_1} and δ_{i_2} transforms $\Gamma_k^{(i)}$ into $-\Gamma_k^{(i)}$. Hence assuming a simplicial subdivision s of M_{2n} such that $\phi(s)$ is a simplicial subdivision of R^n (see M. Richardson (4)), we will have for any simplex σ_k of $\Gamma_k^{(i)}$, $\phi(\sigma_k) = -\phi(T\sigma_k)$. It follows that $\phi(\Gamma_k^{(i)})$ *vanishes identically as a chain on* R^n.

We have thus proved that *any k-cycle on R^n depends on the cycles $\Delta_k^{i_1, i_2, \cdots, i_\beta}$ given by* (1), *the indices $i_1, i_2, \cdots, i_\beta$ being all distinct.* Here the order of the indices affects only the sign of the cycle, and therefore the distinct cycles correspond to the unordered sets of indices.

From now on we shall assume that the cycles δ_i form a canonical set of retrosections of R and that their orientations are such that the following intersection formulas hold:

$$(3) \qquad (\delta_i \cdot \delta_{i+p}) = -(\delta_{i+p} \cdot \delta_i) = +1, \qquad (i = 1, 2, \cdots, p);$$

$$(3') \qquad (\delta_i \cdot \delta_j) = 0, \text{ if } |i - j| \neq p.$$

The intersection numbers of the cycles $\Delta_k^{(i)}$ and $\Delta_{2n-k}^{(j)}$ of complementary dimensions on R^n evidently coincide with those of the cycles $\Gamma_k^{(i)}$ and $\Gamma_{2n-k}^{(j)}$ on the direct product M_{2n}. Using the well-known formulas giving the intersection numbers of cycles on direct products (see Lefschetz (2), p. 243), we find in the present case the following intersection formulas:

$$(4) \qquad (\Delta_k^{i_1, i_2, \cdots, i_\beta} \cdot \Delta_{2n-k}^{j_1, j_2, \cdots, j_{\bar\beta}}) = 0, \text{ if } \beta \neq \bar\beta;$$

$$(5) \quad (\Delta_k^{i_1, i_2, \cdots, i_\beta} \cdot \Delta_{2n-k}^{j_1, j_2, \cdots, j_\beta}) = (-1)^{[\beta(\beta-1)/2]}(\delta_{i_1} \cdot \delta_{j_1})(\delta_{i_2} \cdot \delta_{j_2}) \cdots (\delta_{i_\beta} \cdot \delta_{j_\beta}).$$

From the fact that the intersection numbers are different from zero only for pairs of associated cycles $\Delta_k^{i_1, i_2, \cdots, i_\beta}$ and $\Delta_{2n-k}^{i_1+p, i_2+p, \cdots, i_\beta+p \,(\mathrm{mod}\, 2p)}$, it follows that *the cycles $\Delta_k^{(i)}$ are independent.* Moreover *these cycles form a minimal base for homology* $\approx$, since the above intersection numbers are all 0 or ± 1.

To calculate the Betti number R_k of R^n we observe that the number β of indices in any k-cycle $\Delta_k^{i_1, i_2, \cdots, i_\beta}$ must satisfy the inequalities

$$(6) \qquad \beta \leqq k, \qquad \beta \leqq 2n - k, \qquad \beta \leqq 2p.$$

The first inequality follows from the relation $2\alpha + \beta = k$. The second is

the dual of the first and follows directly by combining the relations $2\alpha + \beta = k$ and $\alpha + \beta + \gamma = n$, getting $2\gamma + \beta = 2n - k$. The third inequality follows from the fact that the indices $i_1, i_2, \cdots, i_\beta$ are distinct integers $\leqq 2p$. We also observe that the difference $k - \beta$ is even. Hence if ρ denotes the smallest of the three integers k, $2n - k$, $2p$, then

$$\left.\begin{aligned} R_k &= 1 + \binom{2p}{2} + \cdots + \binom{2p}{\rho}, \text{ if } k \text{ is even} \\ R_k &= \binom{2p}{1} + \binom{2p}{3} + \cdots + \binom{2p}{\rho}, \text{ if } k \text{ is odd} \end{aligned}\right\} \rho < 2p,$$

and

$$\left.\begin{aligned} R_k &= 1 + \binom{2p}{2} + \cdots + \binom{2p}{2p}, \text{ if } k \text{ is even} \\ R_k &= \binom{2p}{1} + \binom{2p}{3} + \cdots + \binom{2p}{2p-1}, \text{ if } k \text{ is odd} \end{aligned}\right\} \rho = 2p,$$

i. e., if $\rho = 2p$, then $R_k = 2^{2p-1}$ whether k is even or odd.

3. *Linear series $g_n{}^r$ as cycles on R^n. Theorem of Clifford.*[1] *Let R be* the Riemann surface of an algebraic curve f, $f(x, y) = 0$, of genus p. We consider on f a linear series $g_n{}^r$, i. e., a series ∞^r of sets of n points of the curve which are either the sets of level of a linear system of rational functions on f,

$$t_0 + t_1 g_1(x, y) + \cdots + t_r g_r(x, y),$$

or differ from these sets by any number of fixed points. A set of the $g_n{}^r$ is represented by a point of the symmetric product R^n and the $g_n{}^r$ is given on R^n by a certain $2r$-cycle Γ_{2r}. We express this cycle in terms of the cycles $\Delta_{2r}^{(i)}$ of the base:

$$\Gamma_{2r} \sim \epsilon\, \Delta_{2r} + \Sigma\, \epsilon_{i_1, i_2}\, \Delta_{2r}^{i_1, i_2} + \cdots + \Sigma\, \epsilon_{i_1, i_2, \ldots, i_{2\rho}}\, \Delta_{2r}^{i_1, i_2, \ldots, i_{2\rho}}, \tag{7}$$

where $\Delta_{2r} = R^r P_1 P_2 \cdots P_{n-r}$ and where ρ is the smallest of the three numbers p, r, $n - r$ (see the formulas (6), where β and k should be replaced by 2ρ and $2r$ respectively). It should be understood that the above sum contains one and only one term for each unordered set of indices. We agree that the coefficients $\epsilon_{(i)}$ change sign for odd permutations of the indices and are unaltered by even permutations. Then the order of the indices in any cycle $\Delta_{2r}^{(i)}$ is immaterial. For the determination of the coefficients $\epsilon_{(i)}$ we use two properties of a $g_n{}^r$ which follow directly from the definition: a) There exists one and only one set in the $g_n{}^r$ which contains r generic preassigned points of the curve f (*involutorial property*); b) the sets of the $g_n{}^r$ are in

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[2] *Notice to the reader:* **All homologies in this and in the following section are weak homologies (with allowed division).** ***For printing purposes we shall use from now on, without fear of confusion, the symbol $\sim$ instead of $\approx$.***

$(1,1)$ correspondence with the ratios of the parameters t_i, i. e., with the points of a linear complex space S_r (*rationality of the* $g_n{}^r$). From the involutorial property it follows that if $Q_1, Q_2, \cdots, Q_r$ are generic points of f, then Γ_{2r} intersects in one point only the cycle $\Delta_{2(n-r)} = R^{n-r}Q_1Q_2 \cdots Q_r$. Let $Q \equiv (Q_1, Q_2, \cdots, Q_n)$ be the common point of the two cycles and let u_i be a uniformizing parameter on R in the neighborhood of Q_i. Then $u_1, u_2, \cdots, u_n$ are local coördinates of R^n at the point Q and the equations of the loci Γ_{2r} and $\Delta_{2(n-r)}$ in the neighborhood of Q will be respectively:[3]

$$u_{r+j} = f_j(u_1, u_2, \cdots, u_r),$$

$$[j = 1, 2, \cdots, n-r, f_j\text{—a regular function at } (Q_1, Q_2, \cdots, Q_r)];$$

and

$$u_i = \text{const.} \qquad (i = 1, 2, \cdots, r).$$

From these equations it is seen that we are dealing with two analytical varieties whose tangent spaces are independent, and therefore the intersection number of the two cycles is ± 1. Assuming for R and for Γ_{2r} the intrinsic orientation determined by the analytical character of these varieties (Lefschetz (3), Zariski (6), p. 102), we get the sign $+$. Hence $(\Gamma_{2r} \cdot \Delta_{2(n-r)}) = +1$ and since, by (4) and (5), $(\Gamma_{2r} \cdot \Delta_{2(n-r)}) = \epsilon$, it follows $\epsilon = +1$.

We now use the property b). Since the Betti numbers of odd dimensions of S_r all vanish and since Γ_{2r}, as a locus of points, and S_r are homeomorphic, it follows that *the intersection of* Γ_{2r} *with any cycle of odd dimension on* R^n *is* ~ 0 *on* Γ_{2r} *and hence a fortiriori* ~ 0 *on* R^n. *In particular we have the following homologies:*

$$\Gamma_{2r} \cdot \Delta^j{}_{2n-1} \sim 0, \qquad (j = 1, 2, \cdots, 2p). \tag{7'}$$

To make use of these homologies we have only to express the intersection cycles $\Delta^{(i)}_{2r} \cdot \Delta^j{}_{2n-1}$ in terms of the basic cycles $\Delta^{(i)}_{2r-1}$.

We consider the intersection of $\Delta^j{}_{2n-1}$ with a given cycle $\Delta^{i_1, i_2, \cdots, i_{2m}}_{2r}$ and we examine separately two cases, according as j does not or does coincide with one of the numbers $i_1 + p, i_2 + p, \cdots, i_{2m} + p \pmod{2p}$.

1*st case.* $j \not\equiv i_\sigma + p \pmod{2p}$, $(\sigma = 1, 2, \cdots, 2m)$. We have $\Delta^j{}_{2n-1} \sim R^{n-1}\bar{\delta}_j$, where $\bar{\delta}_j$ is a cycle homologous to δ_j and is in generic position with respect to the cycles δ_i. We may assume that the joints $P_1, P_2, \cdots, P_\gamma$ in (1) are distinct and are not on $\bar{\delta}_j$. Since $\bar{\delta}_j$ does not meet the cycles

[3] By the involutorial property of the $g_n{}^r$ it follows that in the neighborhood of its generic set $n-r$ points of the variable set are uniform functions of the remaining r points.

$\delta_{i_1}, \delta_{i_2}, \cdots, \delta_{i_{2m}}$, the only points of $\Delta^{(i)}_{2r}$ which are on $\Delta^j{}_{2n-1}$ are those obtained by converting one of the α arbitrary points relative to the factor R^a into a semifixed point constrained to remain on $\bar{\delta}_j$. If $\alpha = 0$, i. e., if $m = r$, the two cycles have no points in common and hence

$$(8) \qquad \Delta^{i_1 i_2 \cdots i_{2r}}_{2r} \cdot \Delta^j{}_{2n-1} \sim 0.$$

If $m < r$, then the intersection of the two cycles is necessarily a multiple of the cycle

$$R^{a-1}\delta_{i_1}\delta_{i_2} \cdots \delta_{i_{2m}}\bar{\delta}_j P_1 P_2 \cdots P_\gamma \sim \Delta^{i_1, i_2, \cdots, i_{2m}, j}_{2r-1} = \Delta^{(i)j}_{2r-1}.$$

We now compare the indicatrices of the cycles $\Delta^{(i)}_{2r}$, $\Delta^j{}_{2n-1}$ and $\Delta^{(i)j}_{2r-1}$ at a common generic point $P \equiv (P_1, P_2, \cdots, P_\gamma, \cdots, P_n)$, where $P_{\gamma+1}, \cdots, P_{\gamma+a-1}$ are arbitrary points, $P_{\gamma+a}$ is on $\bar{\delta}_j$ and $P_{\gamma+a+\sigma}$ is on δ_{i_σ}. Let (ξ_i, η_i) be an indicatrix of R at the point P_i. We assume that $\xi_{\gamma+a}, \xi_{\gamma+a+1}, \cdots, \xi_n$ coincide with the positive directions of $\bar{\delta}_j, \delta_{i_1}, \cdots, \delta_{i_{2m}}$ respectively. We have then

Indicatrix $\Delta^{(i)}_{2r}$: $(\xi_{\gamma+1}, \eta_{\gamma+1}, \cdots, \xi_{\gamma+a}, \eta_{\gamma+a}, \xi_{\gamma+a+1}, \cdots, \xi_n)$;
Indicatrix $\Delta^j{}_{2n-1}$: $(\xi_1, \eta_1, \cdots, \xi_{\gamma+a-1}, \eta_{\gamma+a-1}, \xi_{\gamma+a}, \xi_{\gamma+a+1}, \eta_{\gamma+a+1}, \cdots, \xi_n, \eta_n)$;
Indicatrix $\Delta^{(i)j}_{2r-1}$: $(\xi_{\gamma+1}, \eta_{\gamma+1}, \cdots, \xi_{\gamma+a-1}, \eta_{\gamma+a-1}, \xi_{\gamma+a+1}, \cdots, \xi_n, \xi_{\gamma+a})$;
Indicatrix R^n : $(\xi_1, \eta_1, \cdots, \xi_n, \eta_n)$,

or also

Indicatrix $\Delta^{(i)}_{2r}$: (Indicatrix $\Delta^{(i)j}_{2r-1}, \eta_{\gamma+a}$);
Indicatrix $\Delta^j{}_{2n-1}$: $(-1)^m$(Indicatrix $\Delta^{(i)j}_{2r-1}, \xi_1, \eta_1, \cdots, \xi_\gamma, \eta_\gamma, \eta_{\gamma+a+1}, \cdots, \eta_n$);
Indicatrix R^n : $(-1)^m$(Indicatrix $\Delta^{(i)j}_{2r-1}, \eta_{\gamma+a}, \xi_1, \eta_1, \cdots, \xi_\gamma, \eta_\gamma, \eta_{\gamma+a+1}, \cdots, \eta_n$),

and consequently

$$(9) \qquad \Delta^{i_1, i_2, \cdots, i_{2m}}_{2r} \cdot \Delta^j{}_{2n-1} \sim + \Delta^{i_1, i_2, \cdots, i_{2m}, j}_{2r-1}, \qquad m < r.$$

Remark. If j equals one of the indices $i_1, \cdots, i_{2m}$, then the intersection of $\Delta^{(i)}_{2r}$ and $\Delta^j{}_{2n-1}$ is ~ 0. In fact, if for instance $j = i_1$, then the interchange of the factors δ_{i_1} and $\bar{\delta}_j$ changes $\Delta^{(i)j}_{2r-1}$ into $-\Delta^{(i)j}_{2r-1}$. On the other hand the new cycle must be homologous to $\Delta^{(i)j}_{2r-1}$, since we have only replaced δ_{i_1} and $\bar{\delta}_j$ by the homologous cycles $\bar{\delta}_j$ and δ_{i_1}. Hence $\Delta^{(i)j}_{2r-1} \sim -\Delta^{(i)j}_{2r-1}$, and consequently $\Delta^{(i)j}_{2r-1} \sim 0$.[4]

[4] One can prove the homology $\Delta^{(i)i_1}_{2r-1} \sim 0$ also directly by observing that if C_2 is a 2-chain on R bounded by $\bar{\delta}_{i_1} - \delta_{i_1}$, then $R^a\delta_{i_1} \cdots \delta_{i_{2m}} C_2 \to \Delta^{(i)i_1}_{2r-1}$.

2nd case. Let j coincide with one of the numbers $i_\sigma + p \pmod{2p}$, say, let $j \equiv i_1 + p$. In addition to the common points which were present in the 1st case, the two cycles $\Delta^{(i)}_{2r}$ and Δ^{j}_{2n-1} have now in common also the points obtained by letting the semifixed point on δ_{i_1} coincide with the common point $P_{\gamma+1}$ of δ_{i_1} and δ_{i_1+p}. Hence the complete intersection of the two cycles consists now, in addition to the cycle $\Delta^{i_1,\cdots,i_{2m},i_1+p}_{2r-1}$ (not present if $m = r$), also of a multiple of the cycle

$$R^a\delta_{i_2}\cdots\delta_{i_{2m}}P_1\cdots P_\gamma P_{\gamma+1} \sim \Delta^{i_2,\cdots,i_{2m}}_{2r-1}.$$

We have therefore

$$\Delta^{i_1,\cdots,i_{2m}}_{2r}\cdot\Delta^{i_1+p}_{2n-1} \sim \epsilon\,\Delta^{i_1,\cdots,i_{2m},i_1+p}_{2r-1} + \eta\,\Delta^{i_2,\cdots,i_{2m}}_{2r-1},$$

where, as in the 1st case, we find $\epsilon = 1$, if $m < r$, and $\epsilon = 0$ if $m = r$. To find η we construct indicatrices at a generic point P of $\Delta^{i_2,\cdots,i_{2m}}_{2r-1}$. Let $P \equiv (P_1,\cdots,P_\gamma,P_{\gamma+1},\cdots,P_n)$, where $P_{\gamma+2}, P_{\gamma+3},\cdots,P_{\gamma+2m}$ are on $\delta_{i_2},\delta_{i_3},\cdots,\delta_{i_{2m}}$ respectively and where $P_{\gamma+2m+1},\cdots,P_n$ are arbitrary points. We fix an indicatrix (ξ_i,η_i) of R at each point P_i, $i \neq \gamma+1$. As for $P_{\gamma+1}$, the common point of δ_{i_1} and δ_{i_1+p}, we take for $\xi_{\gamma+1}$ and $\eta_{\gamma+1}$ the positive directions on δ_{i_1} and δ_{i_1+p} respectively, so that at $P_{\gamma+1}$ the indicatrix is $(\delta_{i_1}\cdot\delta_{i_1+p})(\xi_{\gamma+1},\eta_{\gamma+1})$. We have

Indicatrix $\Delta^{i_1,\cdots,i_{2m}}_{2r}$: $(\xi_{\gamma+1},\cdots,\xi_{\gamma+2m},\xi_{\gamma+2m+1},\eta_{\gamma+2m+1},\cdots,\xi_n,\eta_n)$;
Indicatrix $\Delta^{i_1+p}_{2n-1}$: $(\eta_{\gamma+1},\xi_1,\eta_1,\cdots,\xi_\gamma,\eta_\gamma,\xi_{\gamma+2},\eta_{\gamma+2},\cdots,\xi_n,\eta_n)$:
Indicatrix $\Delta^{i_2,\cdots,i_{2m}}_{2r-1}$: $(\xi_{\gamma+2},\cdots,\xi_{\gamma+2m},\xi_{\gamma+2m+1},\eta_{\gamma+2m+1},\cdots,\xi_n,\eta_n)$;
Indicatrix R^n : $(\delta_{i_1}\cdot\delta_{i_1+p})(\xi_1,\eta_1,\cdots,\xi_n,\eta_n)$;
or
Indicatrix $\Delta^{i_1,\cdots,i_{2m}}_{2r}$: $-$(Indicatrix $\Delta^{i_2,\cdots,i_{2m}}_{2r-1}$, $\xi_{\gamma+1}$);
Indicatrix $\Delta^{i_1+p}_{2n-1}$: $(-1)^m$(Indicatrix $\Delta^{i_2,\cdots,i_{2m}}_{2r-1}$, $\eta_{\gamma+1},\cdots,\eta_{\gamma+2m},\xi_1,\eta_1,\cdots,\xi_\gamma,\eta_\gamma$)
Indicatrix R^n : $(-1)^{m-1}(\delta_{i_1}\cdot\delta_{i_1+p})$
(Indicatrix $\Delta^{i_2,\cdots,i_{2m}}_{2r-1}$, $\xi_{\gamma+1},\eta_{\gamma+1},\eta_{\gamma+2},\cdots,\eta_{\gamma+2m},\xi_1,\eta_1,\cdots,\xi_\gamma,\eta_\gamma$)

and this proves that $\eta = (\delta_{i_1}\cdot\delta_{i_1+p})$. We have thus the following homologies:

$$(10)\quad \Delta^{i_1,\cdots,i_{2m}}_{2r}\cdot\Delta^{i_1+p}_{2n-1} \sim \Delta^{i_1,\cdots,i_{2m},i_1+p}_{2r-1} + (\delta_{i_1}\cdot\delta_{i_1+p})\Delta^{i_2,\cdots,i_{2m}}_{2r-1},\quad m < r;$$

$$(11)\quad \Delta^{i_1,\cdots,i_{2r}}_{2r}\cdot\Delta^{i_1+p}_{2n-1} \sim (\delta_{i_1}\cdot\delta_{i_1+p})\Delta^{i_2,\cdots,i_{2r}}_{2r-1}.$$

Before we apply the homologies (8-11) toward the computation of the coefficients $\epsilon_{(i)}$ in (7), we show that already the form of these homologies

permits us to derive a fundamental property of linear series on an algebraic curve. The coefficients $\epsilon_{(i)}$ do not all vanish ($\epsilon = +1$). Let us assume that there is at least one coefficient with 2σ indices which is different from zero ($0 \leqq \sigma \leqq \rho$), say, let $\epsilon_{j_1, \ldots, j_{2\sigma}} \neq 0$, and that all the coefficients $\epsilon_{(i)}$ with more than 2σ indices vanish. We observe that if $m < r$ then in the expression of $\Delta_{2r}^{i_1, \cdots, i_{2m}} \cdot \Delta^j{}_{2n-1}$ there occurs the cycle $\Delta_{2r-1}^{i_1, \cdots, i_{2m}, j}$ having one more index than $\Delta_{2r}^{i_1, \cdots, i_{2m}}$. Hence, *if we assume that* $\sigma < r$, then we find in the homology (7′) the term $\epsilon_{j_1, \ldots, j_{2\sigma}} \Delta_{2r-1}^{j_1, \cdots, j_{2\sigma}, j}$, and this term obviously does not cancel with any other term. *If, in addition, we assume that* $\sigma < p$, we may choose for j a value distinct from $j_1, j_2, \cdots, j_{2\sigma}$, so that $\Delta_{2r-1}^{j_1, \cdots, j_{2\sigma}, j}$, having distinct indices, is one of the cycles of the minimal base. We thus arrive at a contradiction, since such a cycle cannot occur in the homology (7′) with a coefficient $\neq 0$. It follows that our two assumptions: $\sigma < r$ and $\sigma < p$, cannot be true simultaneously, i. e., *of the two inequalities* $\sigma \geqq r$, $\sigma \geqq p$, *one at least must be true.* Since $\sigma \leqq \rho$ and since ρ is the smallest of the numbers r, p, $n - r$, we conclude that necessarily

$$\sigma = \rho = r \text{ or } p, \tag{12}$$

and, moreover, that *the order* n *and the dimension* r *of a linear series* $g_n{}^r$ *on an algebraic curve of genus* p *satisfy necessarily at least one of the two inequalities:* $n - r \geqq r$, $n - r \geqq p$. The so-called *special series* are those for which the second inequality does not hold, i. e., those for which $r > n - p$. Consequently we may state the above result as follows: *the order* n *and the dimension* r *of a special series* $g_n{}^r$ *always satisfy the inequality* $n \geqq 2r$. This is the *theorem of Clifford* in its usual formulation. Our recognition of the topological character of this classical theorem is well in agreement with the fact that it is by no means an existence theorem, since it gives only an upper limit for the dimension r of a linear series $g_n{}^r$. One cannot expect topological proofs of existence theorems in algebraic geometry!

4. *Computation of the coefficients* $\epsilon_{(i)}$. Using the intersection formulas (8-11) and taking into account (12) and the remark after formula (9), we find for $\Gamma_{2r} \cdot \Delta_{2n-1}^{j+p}$ the following expression:

$$\begin{aligned} \Gamma_{2r} \cdot \Delta_{2n-1}^{j+p} &\sim \sum_{m=0}^{\rho-1} \sum\nolimits_{(i)} \epsilon_{i_1, i_2, \ldots, i_{2m}} \Delta_{2r-1}^{i_1, i_2, \ldots, i_{2m}, j+p} \\ &+ (\delta_j \cdot \delta_{j+p}) \sum_{m=1}^{\rho} \sum\nolimits_{(i)} \epsilon_{j, i_2, \ldots, i_{2m}} \Delta^{i_2, \ldots, i_{2m}} \sim 0. \end{aligned} \tag{13}$$

The first summation is extended to all unordered sets of indices $i_1, \cdots, i_{2m}$

different from $j+p \pmod{2p}$, while the second summation is extended to all unordered sets of indices $i_2, \cdots, i_m$ different from j. Any cycle $\Delta_{2r-1}^{j_1, j_2, \cdots, j_{2m-1}}$ in which the indices are all different from $j+p \pmod{2p}$ occurs in (13) with the coefficient $(\delta_j \cdot \delta_{j+p})\epsilon_{j, j_1, j_2, \ldots, j_{2m-1}}$. Since this holds for $j = 1, 2, \cdots, 2p$, it follows that if the indices are arranged in order of magnitude then the coefficients $\epsilon_{i_1, i_2, \ldots, i_{2m}}$ which are not of the type $\epsilon_{i_1, \ldots, i_m, i_1+p, \ldots, i_m+p}$ are all zero. If we now consider a cycle $\Delta_{2r-1}^{j_1, \cdots, j_{m-1}, j_1+p, \cdots, j_{m-1}+p, j+p}$, we see that its coefficient in (13) equals

$$\epsilon_{j_1, \ldots, j_{m-1}, j_1+p, \ldots, j_{m-1}+p} + (\delta_j \cdot \delta_{j+p})\epsilon_{j, j_1, \ldots, j_{m-1}, j_1+p, \ldots, j_{m-1}+p, j+p}.$$

Hence

$$\epsilon_{i_1, \ldots, i_m, i_1+p, \ldots, i_m+p} = (-1)^m (\delta_{i_1} \cdot \delta_{i_1+p})\epsilon_{i_2, \ldots, i_m, i_2+p, \ldots, i_m+p}.$$

Applying this recurrence relation m times and recalling that $\epsilon = 1$, we obtain

$$\epsilon_{i_1, i_2, \ldots, i_m, i_1+p, i_2+p, \ldots, i_m+p} = (-1)^{[m(m+1)/2]} (\delta_{i_1} \cdot \delta_{i_1+p})(\delta_{i_2} \cdot \delta_{i_2+p}) \cdots (\delta_{i_m} \cdot \delta_{i_m+p}),$$

or, $\epsilon_{i_1, i_1+p, \ldots, i_m, i_m+p} = (-1)^m$, provided $i_1, \cdots, i_m$ are all less than p. *We have therefore the following expression for the cycle Γ_{2r} associated with the g_n^r:*

$$(14) \qquad \Gamma_{2r} \sim \Delta_{2r} - \Sigma\Delta_{2r}^{i_1, i_1+p} + \Sigma\Delta^{i_1, i_1+p, i_2, i_2+p} + \cdots + (-1)^\rho \Sigma\Delta^{i_1, i_1+p, \ldots, i_\rho, i_\rho+p},$$

where the summation indices $i_1, i_2, \cdots$ are less than p and where $\rho = r$ or p according as $r \leqq p$ or $r \geqq p$.

COROLLARY. *The various linear series g_n^r of a given order n and of a given dimension r on a curve f are represented on R^n by homologous cycles (in the sense of homologies with allowed division).*[5]

5. *The Riemann-Roch theorem.* The following are existence theorems and therefore essentially algebraic in nature:

(a) *There exist infinitely many series g_n^r, of increasing orders, such that $r \geqq n-p$.*

(b) *There exists a series g_{2p-2}^{p-1}.*

These theorems follow in an elementary manner from the consideration

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[5] It would be desirable to find out whether R^n does or does not possess torsion, in order to conclude as to the validity of this corollary with respect to homologies without division. It is not true, however, that the linear series g_n^r on f of given order and dimension necessarily form an irreducible algebraic system. This is true only for series of a sufficiently general type, for instance of a sufficiently high order.

of the series cut out on a plane algebraic curve of order m by its adjoint curves of order $l \geqq m-3$. It is found that if $l > m-3$, then the series is of order $2p-2+m(l-m+3)$ and of dimension $\geqq p-2+(l-m+3)$, and if $l = m-3$, then it is of order $2p-2$ and of dimension $\geqq p-1$. We regard theorems (a) and (b) as the algebro-geometric constituents of the Riemann-Roch theorem and we proceed to prove the rest of this theorem topologically. We naturally assume that the topological significance of the two-fold of the genus of an algebraic curve has been already established (for instance, by means of the consideration of the Euler-Poincaré characteristic of the m-sheeted Riemann surface of the curve f). We assume moreover the following two properties of *complete* linear series [6] which follow directly from the definition of linear series by means of linear systems of rational functions on the curve f: 1) two *distinct* complete series of the same order have no sets in common: 2) if a set G of a given complete series $g_{n_2}^{r_2}$ is contained in one or more sets of another complete series $g_{n_1}^{r_1}$ $(n_1 > n_2)$, then the *residual sets* form a *complete* linear series of order $n_1 - n_2$, and this series remains the same as G varies in $g_{n_2}^{r_2}$ (*the residue theorem in its invariantive form*).

(1) *If g_n^r is a special series, then necessarily $n \leqq 2p-2$.* This is a consequence of the theorem of Clifford. In fact, if the g_n^r is special, then $n \geqq 2r \geqq 2(n-p+1)$, i. e., $n \leqq 2p-2$.

From (1) it follows that the series g_n^r of theorem (a) are *complete* and of dimension $r = n-p$, provided $n > 2p-2$. From this and from the fact that these series have an arbitrarily high dimension it follows immediately, by the residue theorem, that

(2) *if g_n^r is any complete series, then $r \geqq n-p$. If $n > 2p-2$, then $r = n-p$.*

For series of order $2p-2$ we now prove the following:

(3) *A series g_{2p-2}^{p-1} is necessarily complete. There cannot exist two distinct series g_{2p-2}^{p-1}.*

The first part of the theorem follows immediately from the theorem of Clifford. To prove the second part of the theorem let us assume that there exist two distinct series g_{2p-2}^{p-1}, and let $\Gamma_{2p-2}^{(1)}$ and $\Gamma_{2p-2}^{(2)}$ be the corresponding cycles on the symmetric product R^{2p-2}. We have from (14):

$$(15) \qquad \Gamma_{2p-2}^{(1)} \sim \Gamma_{2p-2}^{(2)} \sim \Delta_{2p-2} - \Sigma\Delta_{2p-2}^{i_1,\, i_1+p} + \cdots + (-1)^{p-1}\Sigma\Delta_{2p-2}^{i_1,\, i_1+p,\, \ldots,\, i_{p-1},\, i_{p-1}+p}.$$

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[6] I. e., series g_n^r which are not contained in linear series of the same order n and of a higher dimension.

Using the intersection formulas (4), (5) we find:

$$(16)\quad (\Gamma^{(1)}_{2p-2}\cdot\Gamma^{(2)}_{2p-2}) = 1-\binom{p}{1}+\binom{p}{2}+\cdots+(-1)^{p-1}\binom{p}{p-1}=(-1)^{p-1}.$$

The intersection number of the two cycles being different from zero, it follows that the two complete series g^{p-1}_{2p-2} have at least one set in common, and this contradicts the assumption that the two series are distinct, q. e. d.

We have thus proved *the uniqueness of the canonical series* g^{p-1}_{2p-2} and hence also *its invariance under birational transformations.* We observe that by (16) *the virtual degree of the canonical series (as a cycle on* R^{2p-2}*) equals* $(-1)^{p-1}$.

(4) *Any special series* $g_n{}^r$ *is partially contained in the canonical series.*[7]

Proof. By (1) we have necessarily $n \leqq 2p-2$. The case $n = 2p-2$ was already settled in (3), so that we may assume, if we wish, $n < 2p-2$, although the proof below does not require this assumption. It is sufficient to prove the theorem for a series $g_n{}^{n-p+1}$ contained in the given $g_n{}^r$. We consider of the curve f the series of all sets of $2p-2$ points which are made up of a variable set of the $g_n{}^{n-p+1}$ and of a variable set of $2p-2-n$ arbitrary points. This series, which we shall denote by s^{p-1}_{2p-2} is of dimension $p-1$ and is the locus of the linear series $g^{n-p+1}_{2p-2} = g_n{}^{n-p+1} + P_1 + \cdots + P_{2p-2-n}$ as the $2p-2-n$ fixed points P_i of this series vary arbitrarily. We wish to find the cycle Γ_{2p-2} which corresponds to this series on the symmetric product R^{2p-2}, i. e., the expression of this cycle in terms of the basic cycles $\Delta^{(i)}_{2p-2}$. In order not to complicate the notations, we replace the series $g_n{}^{n-p+1}$ by an arbitrary series $g_n{}^r$, we add to the sets of this series k fixed points $P_1, \cdots, P_k$, and we look for the cycle Γ_{2r+2k} which corresponds on R^{n+k} to the algebraic series s^{r+k}_{n+k}, locus of the series $g^r{}_{n+k} = g_n{}^r + P_1 + \cdots + P_k$ as the fixed points P_i of this series vary arbitrarily. Let Γ_{2r} be the cycle which corresponds on R^{n+k} to the series $g^r{}_{n+k}$ and let V_n be the subvariety $R^nP_1P_2\cdots P_k$ of R^{n+k}. As the points P_i vary, the variety V_n varies in an algebraic system $\{V_n\}$ of dimension k. For any preassigned V_n in this system we may assume that the basic cycles $\Delta^{(i)}_{2r}$ of R^{n+k} lie on it, since in the expression of these cycles as symmetric products there occur at least k fixed points. To find the locus of any cycle $\Delta^{(i)}_{2r}$ as the carrying variety V_n varies in $\{V_n\}$, it is only necessary to convert the fixed points $P_1, \cdots, P_k$ into arbitrary points, the effect being that of converting the cycle $\Delta^{(i)}_{2r}$ into the cycle $\Delta^{(i)}_{2r+2k}$ *having the same indices.* However,

[7] I. e., any set of the $g_n{}^r$ is contained in one or more sets of the canonical series. By the residue theorem there exists then the *residual series* $g^{\rho}{}_{2p-2-n}$ of the canonical series with respect to the $g_n{}^r$, where $\rho+1$ is the number of linearly independent canonical sets containing a given set of the $g_n{}^r$.

the actual locus of $\Delta_{2r}^{i_1,\cdots,i_{2m}}$ is $\binom{r+k-m}{k}\Delta_{2r+2k}^{i_1,\cdots,i_{2m}}$, since in the expression of the cycle $\Delta_{2r+2k}^{i_1,\cdots,i_{2m}}$ as a symmetric product there occur $r+k-m$ arbitrary points (i. e. the factor R^{r+k-m}), and any k of these points can be identified with the variable points $P_1,\cdots,P_k$, so that each cell of $\Delta_{2r+2k}^{i_1,\cdots,i_{2m}}$ must be counted $\binom{r+k-m}{k}$ times. Consequently, denoting by Γ^*_{2r} the right-hand member of (14) and by Γ^*_{2r+2k} the locus of the cycle Γ^*_{2r}, we have:

$$(17)\qquad \Gamma^*_{2r+2k} = \binom{r+k}{k}\Delta_{2r+2k} - \binom{r+k-1}{k}\Sigma\Delta_{2r+2k}^{i_1,\, i_1+p} + \cdots + (-1)\binom{r+k-\rho}{k}\Sigma\Delta_{2r+2k}^{i_1,\, i_1+p,\cdots,\, i_\rho,\, i_\rho+p},$$

where $\rho = r$ or p, according as $r \leqq p$ or $r \geqq p$.

Now in the present case it is not difficult to show that *the loci* Γ_{2r+2k}, Γ^*_{2r+2k} *of the homologous cycles* Γ_{2r}, Γ^*_{2r} *are also homologous cycles.* In fact, between any two varieties of the system $\{V_n\}$, say

$$V_n^{(1)} = R^n P_1^{(1)}\cdots P_k^{(1)}, \qquad V_n^{(2)} = R^n P_1^{(2)}\cdots P_k^{(2)},$$

there is the following uniquely determined homeomorphism:

$$(O_1,\cdots,O_n, P_1^{(1)},\cdots,P_k^{(1)}) \leftrightarrow (O_1,\cdots,O_n, P_1^{(2)},\cdots,P_k^{(2)}),$$

and this homeomorphism reduces to the identity if $V_n^{(1)}$ coincides with $V_n^{(2)}$. Since the elements V_n of the system $\{V_n\}$ are in $(1,1)$ correspondence with the points of R^k, it follows that Γ_{2r+2k} and Γ^*_{2r+2k} are singular images on R^{n+k} of the direct products $\Gamma_{2r}\times R^k$ and $\Gamma^*_{2r}\times R^k$. On the other hand, if C_{2r+1} is a chain on V_n bounded by $\Gamma_{2r}-\Gamma^*_{2r}$, then the locus of C_{2r+1} is a singular image of the direct product $C_{2r+1}\times R^k$. Hence locus $C_{2r+1}\rightarrow\Gamma_{2r+2k}-\Gamma^*_{2r+2k}$, and consequently $\Gamma_{2r+2k}\sim\Gamma^*_{2r+2k}$, so that the desired expression of the cycle Γ_{2r+2k} is given by the right-hand member of the homology (17).

In the particular case of the algebraic series s_{2p-2}^{p-1} considered above $(r=n-p+1,\ k=2p-2-n=p-1-r)$, we find for the corresponding cycle Γ_{2p-2}^{p-1} the following homology:

$$\Gamma_{2p-2}\sim\binom{p-1}{p-1-r}\Delta_{2p-2}-\binom{p-2}{p-1-r}\Sigma\Delta_{2p-2}^{i_1,\, i_1+p} + \cdots + (-1)^r\binom{p-1-r}{p-1-r}\Sigma\Delta_{2p-2}^{i_1,\, i_1+p,\cdots,\, i_r,\, i_r+p},$$

where now $\rho = r$, since $n\leqq 2p-2$ and hence $r\leqq p-1$ by the theorem of Clifford.

Let Γ'_{2p-2} be the cycle on R^{2p-2} which corresponds to the canonical series:

$$\Gamma'_{2p-2}\sim\Delta_{2p-2}-\Sigma\Delta_{2p-2}^{i_1,\, i_1+p} + \cdots + (-1)^{p-1}\Sigma\Delta_{2p-2}^{i_1,\, i_1+p,\cdots,\, i_{p-1},\, i_{p-1}+p}.$$

We have

$$(\Gamma_{2p-2}\cdot\Gamma'_{2p-2}) = \binom{p-1}{p-1-r}-\binom{p-2}{p-1-r}\binom{p}{1}+\binom{p-3}{p-1-r}\binom{p}{2}+\cdots+(-1)^r\binom{p}{r}=(-1)^r.$$

Proof. Let $\phi(p, r)$ denote the left-hand member of the above identity. We have $\phi(p, p-1) = 1 - \binom{p}{1} + \binom{p}{2} + \cdots + (-1)^{p-1}\binom{p}{p-1} = (-1)^{p-1}$. Moreover, we have for any $r < p - 1$,

$$\begin{aligned}\phi(p,r) &+ \phi(p, r+1)\\ &= \binom{p}{p-1-r} - \binom{p-1}{p-1-r}\binom{p}{1} + \binom{p-2}{p-1-r}\binom{p}{2} + \cdots + (-1)^{r+1}\binom{p}{r+1}\\ &= \binom{p}{p-1-r}\left[1 - \binom{r+1}{1} + \binom{r+1}{2} - \cdots + (-1)^{r+1}\right] = 0, \quad \text{q. e. d.}\end{aligned}$$

The intersection number $(\Gamma_{2p-2} \cdot \Gamma'_{2p-2})$ being different from zero, it follows that the series s^{p-1}_{2p-2} and the canonical series have at least one set in common, i. e., there exists a set G_n in the g_n^r and there exists a set of $2p-2-n$ points P_i such that $G_n + P_1 + \cdots + P_{2p-2-n}$ is a canonical set, q. e. d.

(5) The above result is essentially equivalent with the Riemann-Roch theorem. For the convenience of the reader we complete the proof. Let g_n^r be a complete special series, $r = n - p + i$, $i > 0$. Adding to the sets of this series $i-1$ arbitrary fixed points, we obtain a series g^r_{n+i-1} which is still special, and hence, by (4), there exists at least one canonical set which contains a set of the g_n^r *and* $i-1$ *arbitrary preassigned points.* As a consequence, *if* j *denotes the number of linearly independent canonical sets containing a given set of the* g_n^r, *then* $j \geqq i$. On the other hand, the residual series of the canonical series with respect to the g_n^r is a g^{j-1}_{2p-2-n}, i. e., a series $g_{n_1}^{n_1-p+i_1}$, where $n_1 = 2p-2-n$ and $i_1 = (r+1) + (j-i) > 0$. Denoting by j_1 the number of linearly independent canonical sets containing a given set of this residual series, we find as above $j_1 \geqq i_1$. Since $j_1 = r+1$, it follows $r+1 \geqq r+1+(j-i)$, i. e., $j \leqq i$, and consequently $j = i$, q. e. d.

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COLLINEATION GROUPS IN A FINITE SPACE WITH A LINEAR AND A QUADRATIC INVARIANT.

By Arthur B. Coble.

Introduction. In the theory of types of regular Cremona transformations in S_k determined by n points it appears [cf. [4], pp. 39-41] that these types are determined by the elements of a linear group with integer coefficients, which is generated by the permutations of $x_1, \cdots, x_n$ and the additional element,

$$(1) \qquad \begin{aligned} A_{1,\ldots,k+1}: x'_i &= x_i + M \qquad & (i = 0, 1, \cdots, k+1), \\ x'_j &= x_j & (j = k+2, \cdots, n). \\ M &= (k-1)x_0 - x_1 - \cdots - x_{k+1}. \end{aligned}$$

This group has a quadratic, and a linear, invariant,

$$(2) \qquad \begin{aligned} Q &= (k-1)x_0^2 - x_1^2 \cdots - x_n^2, \\ L &= (k+1)x_0 - x_1 \cdots - x_n. \end{aligned}$$

It is ordinarily of infinite order.

As generators of this group we may take the $(n-1)$ transpositions, $(x_1x_2), (x_2x_3), \cdots, (x_{n-1}x_n)$, and $A_{1,\ldots,k+1}$. All of these n generators are of period two, and they lie in a conjugate set which ordinarily contains an infinite number of elements. These generating involutions are of the simplest type, harmonic perspectivities with a center q and an S_{n-1} of fixed points, the polar S_{n-1} of q as to Q. Furthermore the centers q of these involutions are *points on* L.

If the elements of this group are reduced mod. p a finite group, $\Gamma(p)$, is obtained, which for prime p can be regarded as a group in a finite space with a quadratic and a linear invariant. The elements which reduce to the identity mod. p constitute an invariant subgroup of the original group whose factor group is $\Gamma(p)$. Thus the structure of the original group is dependent upon that of $\Gamma(p)$.

The groups of linear transformations in a finite field, G. F. $(s = p^m)$, with a quadratic invariant have been studied by Dickson ([1], Chaps. VII, VIII) who, in the determination of their structure, has been led to important series of simple groups. The nature of $\Gamma(p)$ above is dependent however upon its involutorial generators which play no part in Dickson's treatment. We are thus led to consider the nature of the collineation groups with a quadratic

invariant which are generated by these involutions I_q, and the subgroups which arise when the additional linear invariant is introduced. For the most difficult phase of the argument, namely, the simplicity of certain subgroups, we naturally depend upon Dickson, but the general course of the argument is quite distinct from that employed in the *Linear Groups*.

In § 1 certain normal forms of Q appropriate to the geometric treatment are derived which facilitate useful enumerations. In § 2 the collineation groups of Q in the finite S_n are discussed, and their constitution is determined [cf. (10), (13), (16)]. In § 3 the corresponding groups for a quadratic and a linear form, generated by involutions I_q for points q on L, are obtained [cf. (16), (17), (18)].

We make no attempt here to apply these conclusions to the group $\Gamma(p)$ obtained from the Cremona types. Since the groups $\Gamma(2)$ have already been discussed by the author,[5] we consider only the case of an odd prime.

1. Types of proper quadrics in S_n. We are concerned with the geometry of the proper quadric in a finite linear projective space S_n of dimension n which is defined in the Galois field, G. F. $[s = p^m]$, p an odd prime. The number of points in such a space is

$$P_n = (s^{n+1} - 1)/(s - 1). \tag{1}$$

For our purposes we shall usually need to distinguish only two classes of such fields: namely, those for which -1 is a square σ, or a not-square ν. These we denote by

$$\begin{aligned} &\text{F. G. (I)}: \quad -1 = \sigma;\ m \not\equiv 1 \ (\text{mod. } 2) \quad \text{or} \quad p \not\equiv 3 \ (\text{mod. } 4); \\ &\text{F. G. (II)}: \quad -1 = \nu;\ m \equiv 1 \ (\text{mod. } 2) \quad \text{and} \quad p \equiv 3 \ (\text{mod. } 4). \end{aligned} \tag{2}$$

In such a space, and with $p > 2$, the usual theory of harmonic pairs is valid. Thus the customary rational reduction of the proper quadratic form whose discriminant is not zero to a sum of $n + 1$ squares is also valid. We may therefore [cf. also [1], § 168] write the quadric in the form

$$Q(n) \equiv \lambda_0 x_0^2 + \lambda_1 x_1^2 + \cdots + \lambda_n x_n^2 \qquad (\lambda_i \neq 0). \tag{3}$$

Obviously $Q(1)$ contains two, or no, real points according as $-\lambda_0\lambda_1$ is a σ, or a ν. Also $Q(2)$ contains $s + 1$ real points [cf. [1], § 64]. Indeed $Q(2)$ is the usual locus generated by two projective line pencils. Any $Q(n)$ $[n > 2]$ has $Q(2)$ sections obtained by setting all but three of the x's in (3) equal to zero, whence

(4) *Every $Q(n)$, $n \geqq 2$, contains real points and proper bisecants.*

Let y, z be two points of $Q(n)$ whose join yz is a proper bisecant. If y, z are chosen as the last two reference points, and if their respective polar spaces are chosen as $x_n = 0$, $x_{n-1} = 0$, then $Q(n)$ takes the form $Q(n-2) + 2\alpha_{n-1,n}x_{n-1}x_n$. The coefficient $2\alpha_{n-1,n}$ may be removed, or changed at will, by a multiplication. This process can be applied to $Q(n-2)$, and continued until a $Q(0)$ is reached if n is even, or until a $Q(1)$ is reached if n is odd. Hence

(5) *Every proper quadratic form in S_n can be reduced by linear transformation with coefficients in the* G. F. (s) *to one of the following forms:*

$$\begin{aligned} n \textit{ even:}\quad & Q(n) = x_0^2 + x_1x_2 + \cdots + x_{n-1}x_n, \\ & \nu Q(n) = \nu(x_0^2 + x_1x_2 + \cdots + x_{n-1}x_n); \\ n \textit{ odd:}\quad & Q_+(n) = x_0x_1 + x_2x_3 + \cdots + x_{n-1}x_n, \\ & Q_-(n) = -\nu x_0^2 + x_1^2 + x_2x_3 + \cdots + x_{n-1}x_n. \end{aligned}$$

Thus in even spaces there is but one geometric type of quadratic, whereas in odd spaces there are two types. We shall see that the notation is so chosen that $Q_+(n)$ is on more points than $Q_-(n)$.

We shall divide the points x of S_n with respect to $Q(n)$ into *outside points, inside points,* and *quadric points* according as $Q(n)(x) = \sigma, \nu, 0$.

We shall find that, when n is even, the number of outside points of $Q(n)$ is greater than the number of inside points. It is for this reason that we have preferred the form $Q(n)$ over $\nu Q(n)$. For this reason also the form $Q(n)$ cannot be linearly transformed into the form $\nu Q(n)$. On the other hand,

(6) *When n is odd, there exist linear transformations which convert $Q_\pm(n)$ into $\nu Q_\pm(n)$ and thus interchange the inside and outside points of $Q_\pm(n)$.*

Indeed, for the type $Q_+(n)$ an obvious linear transformation is $x_{2i} = \nu' x'_{2i}$, $x_{2i+1} = x'_{2i+1}$ $(i = 0, \cdots, (n-1)/2)$. For the type $Q_-(n)$, a similarly formed transformation takes care of the product terms. If $-1 = \sigma$, the additional equations, $x_0 = \{\nu'/\sigma\nu\}^{1/2}x'_1$, $x_1 = \{\sigma\nu\nu'\}^{1/2}x'_0$, take care of the square terms. However, if $-1 = \nu''$, we have to transform $\nu''\nu x_0^2 + x_1^2$ into $\nu''\nu'\nu x'^2_0 + \nu' x'^2_1$. Dickson [1, § 169] shows that $\nu(x_r^2 + x_s^2)$ can be transformed into $x'^2_r + x'^2_s$, whence $\nu''\nu x_0^2 + x_1^2$ can be transformed into $\nu'(\nu''\nu x'^2_0 + x'^2_1)$. When $Q_\pm(n)(x) = \nu' Q_\pm(n)(x')$, due to the linear transformation, then, if $Q_\pm(n)(x) = \sigma$ or ν, $Q_\pm(n)(x') = \sigma/\nu'$ or ν/ν'; i. e., outside and inside points of Q are interchanged.

Thus, when n is odd, the inside and outside points of $Q_\pm(n)$ play the same geometric rôle with respect to $Q_\pm(n)$. When n is even, they do not [cf. (11)].

In obtaining the canonical form (2), the polar space x'_0 of any point p' not on Q gives rise to a term $\lambda'_0 x'^2_0$. This point p' is conjugate to the first reference point under a collineation which leaves Q unaltered if λ_0, λ'_0 are both squares or both not-squares. Thus all the outside points are conjugate, and all the inside points are conjugate, under the collineation group of Q. On applying a similar argument to the term $x_{n-1}x_n$ of the canonical forms (5), we obtain the theorem:

(7) *Under the collineation group of Q, the outside points of Q, the inside points of Q, the pairs of points (in either order) on Q and on a proper bisecant, and the proper bisecants of Q, each form a conjugate set.*

We compare the canonical forms (5) of Q with the types employed by Dickson. These are

$$(8) \qquad \begin{array}{lll} A: & & y_0^2 + y_1^2 + \cdots + y_n^2, \\ B: & n \text{ odd}: & \nu y_0^2 + y_1^2 + \cdots + y_n^2, \end{array}$$

the type A occurring for all values of n. An obvious transformation converts the product $x_i x_{i+1}$ into $x'^2_i - x'^2_{i+1}$, and this in G. F. (I) into $x'^2_i + x''^2_{i+1}$. Thus each product yields two squares with unit coefficients in G. F. (I), whereas in G. F. (II) it yields two squares with one coefficient ν. As noted above, two squares with coefficients ν, ν' can be converted into two squares with unit coefficients. Hence, in G. F. (I), $Q(n)$ yields the type A. In G. F. (II), $Q(n)$ yields $n/2$ coefficients ν and $\nu Q(n)$ yields $1 + n/2$ coefficients ν. Thus $Q(n)$ is of type A if $n/2$ is even and $\nu Q(n)$ is of type A if $n/2$ is odd. In G. F. (I), $Q_+(n)$ is of type A and $Q_-(n)$ is of type B. In G. F. (II), $Q_+(n)$ is of type A or B according as $(n+1)/2$ is even or odd, while $Q_-(n)$ is of type A or B according as $(n-1)/2$ is even or odd. Hence

(9) *When n is even, type A is $Q(n)$ except in* G. F. (II) *for $n \equiv 2$ (mod. 4) when it is $\nu Q(n)$. When n is odd, types A, B are $Q_+(n)$, $Q_-(n)$ respectively except in* G. F. (II) *for $n \equiv 1$ (mod. 4) when they are $Q_-(n)$, $Q_+(n)$ respectively.*

Let $q(n)$, $o(n)$, $i(n)$ denote the number of points in S_n which are respectively *on, outside, inside* the quadric $Q(n)$, this quadric being one of the three types in (5).

We divide the points of S_n into the following four classes:

(a) $\qquad y_0, y_1, \cdots, y_{n-2}, 0, 0;$

(b) $\qquad y_0, y_1, \cdots, y_{n-2}, 1, 0;$

$\qquad y_0, y_1, \cdots, y_{n-2}, 0, 1;$

(c) $\qquad y_0, y_1, \cdots, y_{n-2}, y, 1;$

(d) $\qquad 0, 0, \cdots, 0, \quad 1, 0;$

$\qquad 0, 0, \cdots, 0, \quad 0, 1;$

$\qquad 0, 0, \cdots, 0, \quad y, 1 \quad (y \neq 0).$

The number of points of type (a) on $Q(n)$ is $q(n-2)$; of each type (b) is $(s-1).\ q(n-2)$, since a factor $\lambda \neq 0$ must be allowed for in y_i; of type (c) is $\{P_{n-2} - q(n-2)\}(s-1)$, since, for each point not on $Q(n-2)$ with any non-zero multiplier λ, y is unique; and of type (d) is two. Thus we have the recursion formula

$$q(n) = (s-1)P_{n-2} + sq(n-2) + 2,$$

with the initial conditions, $q(0) = 0$, $q_+(1) = 2$, $q_-(1) = 0$. This recursion formula is satisfied by the values given in (10).

According to (6), $o_\pm(n) = i_\pm(n)$. Since also $o_\pm(n) + i_\pm(n) + q_\pm(n) = P_n$, and $q_\pm(n)$ has just been determined, the values $o_\pm(n)$, $i_\pm(n)$ must be those given in (10).

There remains the case $Q(n)$, n even. We ask for $o(n)$, the number of points x for which $Q(n)(x)$ is a square $\sigma \neq 0$. The four classes of points above contribute to this number as follows: (a) $o(n-2)$; (b) $2(s-1)o(n-2)$, allowing for a factor $\lambda \neq 0$ in y_i;

(c) $\quad (s-1)\cdot i(n-2)\cdot(s-1)/2 + (s-1)\cdot o(n-2)\cdot(s-3)/2$
$\qquad\qquad + (s-1)\cdot q(n-2)\cdot(s-1)/2$

and (d) $(s-1)/2$. The case (c) needs some explanation. Any $x_0, \cdots, x_{n-2}$ set in Q_{n-2}, after multiplication by $\lambda \neq 0$, yields a ν', σ', 0 in respectively $(s-1)\cdot i(n-2)$, $(s-1)o(n-2)$, $(s-1)q(n-2)$ cases. When ν' occurs we have to pick a $y \neq 0$ for which $\nu' + y = \sigma$. Thus for each of the $(s-1)/2$ squares σ there is a $y \neq 0$. When σ' occurs, in $\sigma' + y = \sigma$ the square $\sigma = \sigma'$ must be avoided to secure $y \neq 0$. On simplifying the total number of points by using $o(n-2) + i(n-2) + q(n-2) = P_{n-2}$, we obtain the recursion formula,

$$o(n) = s\cdot o(n-2) + P_{n-2}\cdot(s-1)^2/2 + (s-1)/2,$$

with the initial value $o(0) = 1$. This yields $o(n)$ in (10), and $i(n)$ in (10) is obtained from $o(n) + i(n) + q(n) = P_n$. Hence the complete enumeration is

$$(10)\quad \begin{aligned} &n\ \textit{even:}\ \ q(n),\ 2o(n),\ 2i(n) = P_{n-1} \qquad\qquad s^n + s^{n/2},\ s^n - s^{n/2},\\ &n\ \textit{odd:}\ \ q_\pm(n), 2o_\pm(n), 2i_\pm(n) = P_{n-1} \pm s^{(n-1)/2},\ s^n \mp s^{(n-1)/2},\ s^n \mp s^{(n-1)/2}. \end{aligned}$$

In order to enumerate the various types of lines in S_n with respect to $Q(n)$ we need the theorem:

(11) *When n is even, the section of* $Q(n)$ *by the polar space of an outside* [*inside*] *point of* $Q(n)$ *is a* $Q_+(n-1)\,[Q_-(n-1)]$. *When n is odd, the section of* $Q_+(n)$ *by the polar space of an outside* [*inside*] *point of* $Q_+(n)$ *is a* $Q(n-1)\,[\nu Q(n-1)]$ *in* G. F. (I), *but a* $\nu Q(n-1)\,[Q(n-1)]$ *in* G. F. (II); *for* $Q_-(n)$, *the sections* $Q(n-1)$ *and* $\nu Q(n-1)$ *are reversed.*

In view of (7) it is sufficient to verify (11) for particular outside and inside points such as $0, \cdots, 0, 1, 1$ and $0, \cdots, 0, 1, \nu$. We observe again the geometric difference between outside and inside points of $Q(n)$, a difference lacking for $Q_\pm(n)$.

A line will be called a *skew line*, a *tangent*, a *secant* (proper bisecant), or a *generator*, of a quadric $Q(n)$ if it meets $Q(n)$ in 0, 1, 2 or more, distinct and real points. Let ρ_0, ρ_1, ρ_2, ρ_3 be the number of these respective lines on a point r of S_n.

Since P_{n-1} is the total number of lines on a point of S_n, we have the relations:

$$(\alpha) \qquad \begin{array}{ll} r \text{ on } Q(n): & \rho_1 + \rho_2 + \rho_3 = P_{n-1}, \\ r \text{ not on } Q(n): & \rho_0 + \rho_1 + \rho_2 = P_{n-1}. \end{array}$$

For any type of quadric, and a quadric point, the number of tangents and generators on the point is P_{n-2}, the number of lines on the point and in its polar space; and the number of generators is $q(n-2)$. Thus $\rho_1 + \rho_3 = P_{n-2}$, $\rho_3 = q(n-2)$ together with (α), yields the first enumeration in (12).

If n is odd, outside and inside points are conjugate under linear transformation. For p a point of either type, ρ_1 is the number, $q(n-1)$, of points in the section of $Q_\pm(n)$ by the polar space of p. The remaining $q_\pm(n) - q(n-1)$ points of $Q_\pm(n)$ are paired on ρ_2 secants. These facts, with (α), yield the second enumeration in (12).

If n is even, on an outside point $\rho_1 = q_+(n-1)$, and

$$2\rho_2 = q(n) - q_+(n-1);$$

on an inside point $\rho_1 = q_-(n-1)$, and $2\rho_2 = q(n) - q_-(n-1)$ [cf. (11)]. Thus, with (α), we have the third enumeration in (12). Hence

(12) *On a point of the quadric*

$$\begin{array}{ll} Q(n): & \rho_1, \rho_2, \rho_3 = s^{n-2} \qquad s^{n-1}, P_{n-3}; \\ Q_+(n): & \rho_1, \rho_2, \rho_3 = s^{n-2} - s^{(n-3)/2}, s^{n-1}, P_{n-3} + s^{(n-3)/2}; \\ Q_-(n): & \rho_1, \rho_2, \rho_3 = s^{n-2} + s^{(n-3)/2}, s^{n-1}, P_{n-3} - s^{(n-3)/2}. \end{array}$$

For a quadric $Q_\pm(n)$, and either an outside or inside point,

$$2\rho_0, 2\rho_1, 2\rho_2 = s^{n-1} \mp s^{(n-1)/2}, 2P_{n-2}, s^{n-1} \pm s^{(n-1)/2}.$$

For a quadric $Q(n)$, and for an

outside point: $2\rho_0, 2\rho_1, 2\rho_2 = s^{n-1} - s^{(n-2)/2}, 2(P_{n-2} + s^{(n-2)/2}), s^{n-1} - s^{(n-2)/2}$;
inside point: $2\rho_0, 2\rho_1, 2\rho_2 = s^{n-1} + s^{(n-2)/2}, 2(P_{n-2} - s^{(n-2)/2}), s^{n-1} + s^{(n-2)/2}$.

We note some further facts employed in the sequel, using the notation p_o and p_i for outside and inside points respectively.

(13) *A secant of Q cuts Q in two points and contains $(s-1)/2$ points p_o and $(s-1)/2$ points p_i. A tangent of Q contains one point of Q and either s points p_o or s points p_i. A skew line of Q contains $(s+1)/2$ points p_o and $(s+1)/2$ points p_i.*

For, if y, z are two points on $Q = (\alpha x)^2$, and yz is a secant, the point $\lambda_1 y + \lambda_2 z$ substituted in Q yields $2\lambda_1\lambda_2(\alpha y)(\alpha z)$. If y, z are not on Q and yz is a skew line of Q, the similar result is $\lambda_1^2(\alpha y)^2 + 2\lambda_1\lambda_2(\alpha y)(\alpha z) + \lambda_2^2(\alpha z)^2$, an imaginary pair in $\lambda_1 : \lambda_2$. In these two cases, for variable $\lambda_1 : \lambda_2$, the theorem follows from (10) for $n = 1$. If y is on Q and z is on a tangent at y we get $\lambda_2^2(\alpha z)^2$ which has the squarity of $(\alpha z)^2$.

(14) *A harmonic pair of $Q_+(1)$ is, in* G. F. (I), *a pair of points p_o, or a pair of points p_i; in* G. F. (II), *a point p_o and a point p_i. A harmonic pair of $Q_-(1)$ is, in* G. F. (I), *a point p_o and a point p_i; in* G. F. (II) *a pair of points p_o or a pair of points p_i.*

For, with $Q_+(1) = x_0 x_1$, the harmonic pair $x_0 : x_1$, $-x_1 : x_0$ yields $x_0 x_1$, $-x_0 x_1$, which have the same squarity in G. F. (I) but not in G. F. (II). With $Q_-(1) = -\nu x_0^2 + x_1^2$, the harmonic pair $x_0 : x_1$, $x_1 : \nu x_0$ yields $-\nu x_0^2 + x_1^2$, $-\nu x_1^2 + \nu^2 x_0$, which differ by the factor $-\nu$, a not-square in G. F. (I), a square in G. F. (II).

2. The collineation groups of the quadrics $Q(n)$ in S_n. We wish to determine first the order of the group of collineations which leaves the quadric $Q(n)$ in S_n unaltered, i. e., the collineations whose linear transformations leave the form $Q(n)$ invariant to within a factor. We denote this order generically by $N(n)$, or, more specifically, by $N(n)$ [n even]; $N_+(n)$, $N_-(n)$ [n odd].

According to **1** (10), (12) the number of secants of $Q(n)$ is $q(n) \cdot s^{n-1}/2$.

The number of ordered pairs of points of $Q(n)$ on a secant is $q(n)\cdot s^{n-1}$. Two such points being taken as the last two reference points, and their polar spaces as x_n, x_{n-1} respectively, $Q(n)$ takes the form,

$$Q(n-2)+2a_{n-1,n}x_{n-1}x_n.$$

Any collineation which leaves the last two reference points unaltered, and which leaves $Q(n)$ unaltered to within a factor, is made up of one of the $N(n-2)$ collineations which converts $Q(n-2)$ into $\lambda Q(n-2)$, and of $x_{n-1}=\lambda_{n-1}x'_{n-1}$, $x_n=\lambda_n x'_n$, where $\lambda_{n-1}\lambda_n=\lambda$. This furnishes $s-1$ choices for $\lambda_{n-1}\lambda_n$, whence

$$(1)\qquad N(n)=(s-1)\cdot s^{n-1}\cdot q(n)\cdot N(n-2).$$

To obtain initial conditions we observe that $Q(2)$, a conic, contains $s+1$ real points, and that a collineation of $Q(2)$ is obtained from three corresponding pairs, whence $N(2)=(s+1)s(s-1)=(s^2-1)s$. A collineation which leaves $Q_+(1)$, two real points on a line, unaltered must leave each point unaltered or must interchange them (two choices), and must send a third point into any one of $s-1$ places, whence $N_+(1)=2(s-1)$. Similarly, in the extended field determined by the imaginary pair $Q_-(1)$, $N_-(1)=2(s+1)$. Thus we have the conditions,

$$(2)\qquad N(2)=(s^2-1)s,\qquad N_\pm(1)=2(s\mp 1).$$

For n even, $q(n)\cdot(s-1)=P_{n-1}\cdot(s-1)$ [cf. **1** (10)] $=s^n-1$, whence

$$N(n)=(s^n-1)s^{n-1}(s^{n-2}-1)s^{n-3}\cdots(s^2-1)s.$$

For n odd,

$$q_\pm(n)\cdot(s-1)=\{P_{n-1}\pm s^{(n-1)/2}\}(s-1)=(s^{(n-1)/2}\pm 1)(s^{(n+1)/2}\mp 1).$$

This leads to

$$\begin{aligned}N_\pm(n)&=(s^{(n+1)/2}\mp 1)(s^{(n-1)/2}\pm 1)s^{n-1}\cdot N_\pm(n-2)\\&=(s^{(n+1)/2}\mp 1)(s^{(n-1)/2}\pm 1)s^{n-1}\cdot(s^{(n-1)/2}\mp 1)(s^{(n-3)/2}\pm 1)s^{n-3}\cdot N_\pm(n-4)\\&=(s^{(n+1)/2}\mp 1)s^{n-1}(s^{n-1}-1)s^{n-3}\cdot(s^{(n-3)/2}\pm 1)N_\pm(n-4).\end{aligned}$$

Recalling the values (2) for $N_\pm(1)$, we find that

(3) *The order of the collineation group of the quadric is*

n even: $N(n)=(s^n-1)s^{n-1}(s^{n-2}-1)s^{n-3}\cdots(s^2-1)s;$
n odd: $N_\pm(n)=2(s^{(n+1)/2}\mp 1)s^{n-1}(s^{n-1}-1)s^{n-3}(s^{n-3}-1)\cdots s^2(s^2-1).$

If p is any point not on the quadric $Q=(\alpha x)^2$, we denote by I_p the perspective involution with center p, and linear space of fixed points π, π being the polar space of p as to Q. The equations of this involution are

$$(4) \qquad x' = x - p \cdot 2(\alpha p)(\alpha x)/(\alpha p)^2.$$

A set of n linearly independent points in $\pi = (\alpha p)(\alpha x) = 0$ are each fixed under (4) with respective multipliers $+1$, and p is fixed with multiplier -1, whence I_p is an involution with determinant -1. For it Q is an absolute invariant, i. e.,

$$(5) \qquad (\alpha x')^2 = (\alpha x)^2.$$

We seek now to determine the collineation groups generated by these involutions I_p. For this some lemmas are necessary.

(6) *The points of Q are conjugate under sequences of involutions I_p.*

Let a, a' be any two distinct points of Q. If aa' is a secant and p is a further point on this secant, then I_p sends a into a'. If aa' is a generator, let b be a point of Q not on the polar space π_a of a. The points of Q not on π_a do not themselves lie in a linear space since Q is not a pair of such spaces. Hence the polar space $\pi_{a'}$ will not exhaust the points b, i. e., points b exist such that ab, $a'b$ are secants. If p, q are points not on Q but on these respective secants, then I_pI_q sends a into a'.

(7) *The ordered pairs of points on Q on secant lines are conjugate under sequences of involutions I_p.*

Let a, b and a', b' be two such ordered pairs. We first send a into a' by a sequence of I_p's, b then going into b''. The plane $a'b''b'$ meets Q in a conic K since $a'b''$ and $a'b'$ are secants and $a'b''b'$ is a proper triangle. If $b'b''$ is a secant of K which meets the tangent $\pi_{a'}$ in p, then I_p leaves a' unaltered and sends b'' into b'. If $b'b''$ is a generator, K is made up of $b'b''$ and a line on a' which meets $b'b''$ in f. If $\bar{b}$ is a third point on $a'f$, and q a third point on $\bar{b}b'$, such that qa' meets $b''\bar{b}$ at p, then I_pI_q sends a' into itself and sends b'' into b'.

Let now T_n be any collineation which leaves $Q(n)$ unaltered. Let a_{n-1}, a_n be the poles of x_n, x_{n-1} respectively, and let T_n send a_{n-1}, a_n into a'_{n-1}, a'_n. According to (7) there is a product, Π_{n-1}, of involutions I_p which sends a'_{n-1}, a'_n into a_{n-1}, a_n and leaves $Q(n)$ unaltered. Then $T_{n-2} = T_n\Pi_{n-1}$ is a collineation which leaves $Q(n)$, a_{n-1}, a_n each unaltered. Hence T_{n-2} is a

collineation on the variables $x_0, \cdots, x_{n-2}$ alone which leaves $Q(n-2)$ unaltered combined with the multiplication $x_{n-1} = \lambda_{n-1}x'_{n-1}$, $x_n = \lambda_n x'_n$.

If n is even, this process can be continued until a transformation of the form,

$$T_n\Pi(I_p) \; : \; x_0 = \lambda_0 x'_0, \, x_1 = \lambda_1 x'_1, \cdots, x_n = \lambda_n x'_n,$$

is obtained, where

$$\lambda_0^2 = \lambda_1\lambda_2 = \cdots = \lambda_{n-1}\lambda_n.$$

Dividing through by λ_0, the conditions on the new multipliers are

$$1 = \lambda_1\lambda_2 = \cdots = \lambda_{n-1}\lambda_n.$$

But $x_1 = \lambda_1 x'_1$, $x_2 = \lambda_2 x'_2$ $(\lambda_1\lambda_2 = 1)$ is the product of $x_1 = \lambda_1 x'_2$, $x_2 = \lambda_1^{-1}x'_1$ and $x_1' = x_2''$, $x_2' = x_1''$, the other variables being unaltered. These two factors are involutions I_p, whence $T_n\Pi(I_p) = \Pi'(I_p)$, or $T_n = \Pi''(I_p)$.

In the case n odd for $Q_+(n)$ we can find $\Pi(I_p)$ such that $T_n\Pi(I_p)$ is $x_i = \lambda_i x'_i$ $(i = 0, \cdots, n)$, where

$$\lambda_0\lambda_1 = \lambda_2\lambda_3 = \cdots = \lambda_{n-1}\lambda_n = \mu.$$

Case (a). If $\mu = \epsilon^{2l}$ (ϵ a primitive root in G. F.) and the multipliers be divided by ϵ^l we have as before that $T_n\Pi(I_p) = \Pi'(I_p)$ and $T_n = \Pi''(I_p)$.

Case (b). If $\mu = \epsilon^{2l+1}$, and the multipliers be divided by ϵ^l we get $T_n\Pi(I_p) = \tau\Pi'(I_p)$, where

$$\tau: \quad x_{2i} = \epsilon x'_{2i}, \; x_{2i+1} = x'_{2i+1} \qquad [i = 0, \cdots, (n-1)/2].$$

Then $T_n = \tau\Pi''(I_p)$. Since I_p leaves $Q_+(n)$ absolutely unaltered, and τ reproduces it multiplied by ϵ, I_p does not interchange outside and inside points, whereas τ does.

In the case of $Q_-(n)$ we can find $\Pi(I_p)$ such that $T_n\Pi(I_p)$ is

$$x_0 = \alpha x'_0 + \beta x'_1, \; x_1 = \gamma x_0 + \delta x'_0, \; x_i = \lambda_i x'_i \qquad (i = 2, \cdots, n),$$

where

$$\lambda_2\lambda_3 = \lambda_4\lambda_5 = \cdots = \lambda_{n-1}\lambda_n = \mu,$$

and

$$(-\nu x_0^2 + x_1^2) = \mu(-\nu x_0'^2 + x_1'^2).$$

The involutorial elements which carry $-\nu x_0^2 + x_1^2$ into a multiple of itself are the reflections in the members of the pencil (variable ρ) of quadratic forms, $\nu x_0^2 + x_1^2 + \rho x_0 x_1$, apolar to $-\nu x_0^2 + x_1^2$. The discriminant, $\Delta = \rho^2 - 4\nu$, of a member is not zero in G. F. The $s + 1$ values $x_0 : x_1$ divide into $(s + 1)/2$

real apolar pairs $(\Delta = \sigma)$, and the remaining $(s+1)/2$ members have imaginary roots $(\Delta = \nu')$. The polarized quadratic yields the reflection, $x'_0 = \rho x_0 + 2x_1, x'_1 = -2\nu x_0 - \rho x_1$, for which $-\nu x'^2_0 + x'^2_1 = \Delta[-\nu x_0^2 + x_1^2]$. The reflections in the real pairs generate a dihedral collineation g_{s+1}, for which factors of proportionality may be so chosen that $-\nu x_0^2 + x_1^2$ is absolutely unaltered; the reflections in the imaginary pairs change $-\nu x_0^2 + x_1^2$ into $\nu'(-\nu x'^2_0 + x'^2_1)$. Hence

Case (a). If μ is a square, $T_n \Pi(I_p) = \Pi'(I_p)$, or $T_n = \Pi''(I_p)$.

Case (b). If μ is a not-square, $T_n \Pi(I_p) = \tau' \Pi'(I_p)$, where τ' is the collineation in $x_0 : x_1$ above augmented by

$$x_{2i} = \mu x'_{2i},\ x_{2i+1} = x'_{2i+1} \qquad [i = 1, \cdots, (n-1)/2].$$

Hence

(8) *The collineation group generated by involutions I_p are, for n even, the entire collineation group of $Q(n)$ of order $N(n)$; and, for n odd, those invariant subgroups of the collineation groups of $Q_\pm(n)$ of index 2 and orders $N_\pm(n)/2$, which do not interchange the inside and outside points of $Q_\pm(n)$.*

We shall denote these groups generated by involutions I_p by $G(n)(I_p)$, $G_+(n)(I_p)$, $G_-(n)(I_p)$ respectively. We seek now to determine their structure.

An involution I_p effects an even or an odd permutation of the points of Q according as ρ_2 for the point p is even or odd. We examine then the parity of ρ_2 as given in **1** (12) and find that:

(9) *The parity of ρ_2 for a point p_o or a point p_i with respect to Q is given in the table:*

	$Q(n)$			$Q_+(n)$	$Q_-(n)$
(mod. 4)	p_o	p_i	(mod. 4)	p_o, p_i	p_o, p_i
$n \equiv 0$	*even*	*odd*	$n \equiv 1$	*odd*	*even*
$n \equiv 2$			$n \equiv 3$		
G. F. (I)	*even*	*odd*	G. F. (I)	*odd*	*even*
G. F. (II)	*odd*	*even*	G. F. (II)	*even*	*odd.*

We have seen in (8) that for n even the collineation group of $Q(n)$ is generated by involutions I_p. According to (9) some of these effect odd permutations of the points of $Q(n)$, whence $G(n)(I_p)$ has an invariant subgroup of index 2. The points p for which the I_p are even are points p_o except when $n \equiv 2 \pmod 4$ in G. F. (II). But this exceptional case is precisely that in which $\nu Q(n) = A$, or $Q(n) = \nu' A$ [cf. **1** (9)]. If this factor ν' is removed the points p_i become points p_o. Hence

(10) *The collineation group of the quadric,*

$$A \equiv y_0^2 + y_1^2 + \cdots + y_n^2 \ (n \text{ even}),$$

of order $N(n)$ *is generated by involutions* I_p [cf. (8)]. *It contains a simple invariant subgroup of index two and order* $N(n)/2 = FO(n+1, p^m)$, *which is generated by involutions* I_{po}.

In connection with the proof of this theorem we observe that the notation $FO(n+1, p^m)$ is that of Dickson ([1], p. 191). The simplicity of the invariant subgroup of this order (except in the case $n=2$, $p^m=3$, when it is the even G_{12} on the four points of the conic $y_0^2 + y_1^2 + y_2^2$) is proved by Dickson. Dickson's groups of linear transformations of determinant unity which leave Q absolutely unaltered must have series of composition whose indices coincide with those of our collineation groups, except for an index 2 when n is odd due to the factor of proportionality ± 1, or except for indices (factors of $s-1$) which arise from the fact that our collineations do not leave Q absolutely unaltered. Since none of these exceptional indices could have the value $FO(n+1, p^m)$, the simplicity of the collineation group here obtained must follow. That the group is generated by involutions I_{po} is due to the fact that these involutions must generate a subgroup invariant under the group of order $N(n)$. These considerations apply also in the demonstration of (13) and (16).

For odd n the quadrics $Q_\pm(n)$ contain systems of linear spaces $S_{(n-1)/2}$ which in the case of $Q_+(n)$ are real; of $Q_-(n)$, are conjugate imaginary. We recall the theorem of C. Segre [2] [cf. also Bertini [3]]:

(11) *If n is odd, a proper quadric in S_n contains two systems of linear spaces $S_{(n-1)/2}$. If $n \equiv 1$ mod. 4 $\{n \equiv 3$ mod. 4$\}$, two $S_{(n-1)/2}$'s belong to the same or different systems according as they have an S_{2d} $\{S_{2d-1}\}$ or an S_{2d-1} $\{S_{2d}\}$ in common* $[2d, 2d-1 < (n-1)/2]$.

In the case of $Q_+(n)$ one such $S_{(n-1)/2}$ on $Q_+(n)$ is $x_0 = x_2 = \cdots = x_{n-1} = 0$. This is transformed by the I_p which interchanges x_0, x_1 into

$$x_1 = x_2 = x_4 = \cdots = x_{n-1} = 0.$$

These two $S_{(n-1)/2}$'s meet in the $S_{(n-3)/2}$,

$$x_0 = x_1 = x_2 = x_4 = \cdots = x_{n-1} = 0.$$

If $n \equiv 1$ mod. 4, $(n-3)/2 = 2d-1$. If $n \equiv 3$ *mod.* 4, $(n-3)/2 = 2d$. In either case, the two $S_{(n-1)/2}$'s belong to different systems. By continuous

variation this interchange takes place throughout the systems. Since all points p are conjugate under the group of $Q_+(n)$, each of the involutions I_p interchanges the two systems.

In the case of $Q_-(n)$, the $S_{(n-1)/2}$ given by

$$\sqrt{\nu}\, x_0 + x_1 = x_2 = x_4 = \cdots = x_{n-1} = 0$$

is transformed by the I_p which changes the sign of x_0 into its conjugate imaginary. These two $S_{(n-1)/2}$'s meet in the same $S_{(n-3)/2}$ as before, and again belong to different systems. Hence

(12) *The groups $G_\pm(n)(I_p)$ of the quadrics $Q_\pm(n)$ each contain an invariant subgroup of index two, generated by an even number of I_p's, which leaves each of the two systems of $S_{(n-1)/2}$'s (real or imaginary) on $Q_\pm(n)$ unaltered.*

This enables us to give the complete constitution of the group of the quadric of Dickson's type B:

(13) *The quadric,*

$$B = \nu y_0{}^2 + y_1{}^2 + \cdots + y_n{}^2 \ (n \text{ odd}),$$

[a $Q_+(n)$ when $n \equiv 1$ mod. 4 in G. F. (II), *otherwise a $Q_-(n)$] has a collineation group of order $N_+(n)$ or $N_-(n)$, as the case may be [cf. (3)]. This has an invariant subgroup $G(I_p)$ of index two, generated by elements I_p, whose elements transform points p_o, p_i into points p_o, p_i respectively. This subgroup has an invariant subgroup of index two, generated by an even number of elements I_p, whose elements transform each of the two systems of $S_{(n-1)/2}$'s on the quadric into itself. Thus the group of B has factors of composition,*

$$2,\ 2,\ N_\pm(n)/4 = SO(n+1, p^m) \qquad [cf.\ ^1, \text{ p. } 191].$$

There remains only Dickson's type A, n odd, which is a $Q_-(n)$ when $n \equiv 1$ mod. 4 in G. F. (II) but which otherwise is a $Q_+(n)$. For these cases we see in (9) that ρ_2 is odd in G. F. (I) and even in G. F. (II).

Though the table (9) indicates invariant subgroups of the groups $G_\pm(I_p)$ in certain cases, it does not completely describe either the type A or the type B. For this purpose we seek the number of conjugate o-pairs and i-pairs under I_{p_o} and under I_{p_i}. Consider then I_{p_o} for the quadric Q. On p_o there are ρ_1 tangents, ρ_2 secants, and ρ_0 skew-lines of Q. On a tangent through p_0 the point p_o is fixed and the contact on Q is fixed. The $s-1$ other points are all o-points [cf. 1 (13)] and yield $(s-1)/2$ o-pairs. On a secant through p_o

the fourth harmonic of p_o as to the two points on Q is a point p'_o in G. F. (I) and a point p_i in G. F. (II) *cf.* [**1** (14)]. Hence [cf. **1** (13)] the secant contains $(s-5)/4$ o-pairs and $(s-1)/4$ i-pairs in G. F. (I), and $(s-3)/4$ o-pairs and $(S-3)/4$ i-pairs in G. F. (II). On a skew line through p_o the fourth harmonic of p_o as to the two imaginary points on Q is a point p_i in G. F. (I), and a point p_o in G. F. (II). Hence the skew line contains $(s-1)/4$ o-pairs and $(s-1)/4$ i-pairs in G. F. (I), and $(s-3)/4$ o-pairs and $(s+1)/4$ i-pairs in G. F. (II). Using a similar argument for I_{p_i}, we have the following result:

(14) *The number of pairs of conjugate o-points, and of conjugate i-points of the involution I_{p_o}, and the involution I_{p_i}, for a quadric Q is given by the table:*

$$
\begin{array}{llrl}
I_{p_o}: & \text{G. F. (I)}: & \rho_1(s-1)/2+\rho_2(s-5)/4+\rho_0(s-1)/4 & o\text{-pairs},\\
 & & \rho_2(s-1)/4+\rho_0(s-1)/4 & i\text{-pairs},\\
 & \text{G. F. (II)}: & \rho_1(s-1)/2+\rho_2(s-3)/4+\rho_0(s-3)/4 & o\text{-pairs},\\
 & & \rho_2(s-3)/4+\rho_0(s+1)/4 & i\text{-pairs},\\
I_{p_i}: & \text{G. F. (I)}: & \rho_2(s-1)/4+\rho_0(s-1)/4 & o\text{-pairs},\\
 & & \rho_1(s-1)/2+\rho_2(s-5)/4+\rho_0(s-1)/4 & i\text{-pairs},\\
 & \text{G. F. (II)}: & \rho_2(s-3)/4+\rho_0(s+1)/4 & o\text{-pairs},\\
 & & \rho_1(s-1)/2+\rho_2(s-3)/4+\rho_0(s-3)/4 & i\text{-pairs},
\end{array}
$$

where the numbers ρ_0, ρ_1, ρ_2 refer to the point p_o or p_i in I_p, and for given Q are obtained from **1** (12).

We are interested only in the parity of the numbers of the table (14). For $Q_\pm(n)$, $\rho_1 = P_{n-2} = s^{n-2} + \cdots + s + 1$. Since s and n are odd, ρ_1 is even. Since $(s-1)/2$ is integral, $\rho_1(s-1)/2$ is even and may be dropped in (14) without affecting parity. By adding and subtracting ρ_2 in the first, fourth, sixth, and seventh of the above eight numbers the factor $\rho_2+\rho_0 = s^{n-1}$ appears in all. This being odd, it may be replaced by unity. The ρ_2 still remaining may be replaced by ± 1 in G. F. (I), and may be dropped in G. F. (II). Thus the parity of the eight numbers in (14) is that of

$$
(15)\quad
\begin{array}{lll}
I_{p_o}: \text{G. F. (I)}: & (s-1)/4+1 & I_{p_i}: \text{G. F. (I)}: (s-1)/4\\
 & (s-1)/4 & \qquad\qquad (s-1)/4+1\\
\text{G. F. (II)}: & (s-3)/4 & \quad : \text{G. F. (II)}: (s-3)/4+1\\
 & (s-3)/4+1 & \qquad\qquad (s-3)/4.
\end{array}
$$

We observe first that if the number of o-pairs is odd, the number of i-pairs is even and vice-versa. We observe also that the parity of any one of the

four numbers for I_{p_0} is opposite to that of the corresponding number for I_{p_i}. Hence

(16) *The quadric,*

$$A = y_0^2 + y_1^2 + \cdot \cdot \cdot + y_n^2 \ (n \text{ odd}),$$

[*a* $Q_-(n)$ *when* $n \equiv 1$ *mod.* 4 *in* G. F. (II), *otherwise a* $Q_+(n)$] *has a collineation group of order* $N_+(n)$ *or* $N_-(n)$ *as the case may be* [*cf.* (3)]. *This has an invariant subgroup* $G(I_p)$ *of index two, generated by elements* I_p, *whose elements transform points* p_o, p_i *into points* p_o, p_i *respectively. This subgroup has an invariant subgroup of index two, generated by an even number of elements* I_p, *whose elements transform each of the two systems of* $S_{(n-1)/2}$'s *on the quadric into itself. This second invariant subgroup has an invariant subgroup of index two, generated by pairs of involutions* I_{p_0}, *or by pairs* I_{p_i}, *whose elements permute both the outside points and the inside points of the quadric evenly. Thus the group of A has factors of composition,* 2, 2, 2,

$$N_\pm(n)/8 = FO(n+1, p^m) \qquad [cf.\ ^1, \text{p. } 191].$$

3. The groups defined by a quadric Q and a linear space L. The space L being the polar of a point p, we shall have to distinguish the three cases in which p is a q-, an o-, or an i-point, and correspondingly L is an L_q, L_o, or L_i space. The order of the collineation group of Q, L is the order of the group of Q divided by the number of points of the kind in question, these values being obtained from **2** (3) and **1** (10) respectively. Thus we find that

$$
\begin{aligned}
(1) \qquad &\text{I 1}: & [Q(n), L_q] &= (s-1)s^{n-1} \cdot N(n-2),\\
&\text{I 2}: & [Q(n), L_o] &= N_+(n-1),\\
&\text{I 3}: & [Q(n), L_i] &= N_-(n-1),\\
&\text{II 1}: & [Q_+(n), L_q] &= (s-1)s^{n-1} \cdot N_+(n-2),\\
&\text{II 2}: & [Q_+(n), L_{o,i}] &= 4N(n-1),\\
&\text{III 1}: & [Q_-(n), L_q] &= (s-1)s^{n-1} \cdot N_-(n-2),\\
&\text{III 2}: & [Q_-(n), L_{o,i}] &= 4N(n-1).
\end{aligned}
$$

In the case of the quadrics we shall be concerned primarily with the groups $G(I_p)$ generated by the I_p's. For these the cases II, III above become

$$
\begin{aligned}
(2) \qquad & [G_\pm(n)(I_p), L_q] = (s-1)s^{n-1} \cdot G_\pm(n-2)(I_p),\\
& [G_\pm(n)(I_p), L_{o,i}] = 2N(n-1).
\end{aligned}
$$

We first examine the section of Q, a quadric of any one of the three types, by one of its tangent spaces L_q, tangent at q. If q is the next to the last

reference point, a_{n-1}, the space L_q is $x_n = 0$. This space cuts Q in a quadric in the S_{n-1}, $x_n = 0$, with a node at a_{n-1}. This quadric is a point section by a_{n-1} of a proper quadric $Q(n-2)$ in an S_{n-2} in the S_{n-1} but not on a_{n-1}. We may take this to be the S_{n-2}, $x_{n-1} = x_n = 0$. Then $Q(n-2) = Q(x_{n-1} = x_n = 0)$ is in the standard form and is of the same type as Q. Let $Q = (\alpha x)^2$ and $Q(n-2) = (\beta x)^2 = (\alpha x)^2[x_{n-1} = x_n = 0]$.

The lines λ on q and in L_q give rise to the points in S_{n-2}. These lines λ are the tangents and generators of Q on q. The generators give rise to the points of $Q(n-2)$, the tangents to the o- or i-points of $Q(n-2)$ according as to whether outside q they contain only o-points or only i-points of Q [*cf.* **1** (13)]. The collineation group of $[Q, L_q]$ permutes these lines λ. Those collineations of the group which leave each line λ unaltered [the identity in the group of $Q(n-2)$] must form an invariant subgroup of $[Q, L_q]$. The existence of the factor $N(n-2)$ in the order $[Q, L_q]$ suggests that this invariant subgroup has the order $(s-1)s^{n-1}$. We consider the form of the elements,

$$x'_i = \Sigma_j \gamma_{ij} x_j \qquad (i, j = 0, \cdots, n), \tag{3}$$

of this invariant subgroup.

Let the coefficients in the last column of (3) be $y_0, y_1, \cdots, y_n$. Since $Q = (\alpha x)^2$ is to be unaltered, the term in x_n^2 in $(\alpha x')^2$ must not appear, i.e., $(\alpha y)^2 = 0$. Since the lines λ on a_{n-1} are to be invariant, only the $(k+1)$-th and n-th coördinates of the $(k+1)$-th reference point can be affected whence in the $(k+1)$-th column only the coefficients $\gamma_{k,k}$, $\gamma_{n-1,k}$ appear $(k = 0, \cdots, n-2)$. Since also the point a_{n-1} is to be invariant, $\gamma_{n-1,n-1}$ is the only non-zero coefficient in the n-th column. Since the point $1, 1, \cdots, 1, 0, 0$ is to go into $1, 1, \cdots, 1, \mu, 0$, $\gamma_{00} = \gamma_{11} = \cdots = \gamma_{n-2,n-2}$. The determinant of (3) is the product of these γ's and $\gamma_{n-1,n-1} y_n$, whence $y_n \neq 0$, and we may take $\gamma_{00} = \cdots = \gamma_{n-2,n-2} = 1$. Since $(\alpha x')^2 = \rho(\alpha x)^2$, we find from the leading term, x_0^2, $x_0 x_1$ or $-\nu x_0^2$, that $\rho = 1$, and then, from the term in $x_{n-1}x_n$, that $\gamma_{n-1,n-1} = 1/y_n$. Again from the absolute invariance of Q we find that

$$\gamma_{n-1,0}x_0 + \cdots + \gamma_{n-1}x_{n-2} = -(y\partial/\partial x)Q(n-2)/y_n. \tag{4}$$

The point y, on Q but not on $x_n = 0$, is joined to a_{n-1} by one of the s^{n-1} secants of Q on a_{n-1} [cf. **1** (12)]. Hence

(5) *For any of the s^{n-1} choices of y on Q but not on L_q $(q = a_{n-1}, L_q \equiv x_n = 0)$, and for any non-zero factor of proportionality y_n [$(s-1)$ choices], the ele-*

ment (3) *is a collineation of determinant unity which transforms Q into itself and leaves every line λ on q in L_q unaltered if the coefficients satisfy the requirements: the matrix of the first $n-1$ rows and columns is the unit matrix, all the other γ_{ij}'s are zero except that $\gamma_{n-1,n-1}=1/y_n$, the last column is $y_0,\cdots,y_n$, and the coefficients $\gamma_{n-1,0},\cdots,\gamma_{n-1,n-2}$ are as in* (4). *These* $(s-1)s^{n-1}$ *elements form an invariant subgroup $g_{(s-1)s^{n-1}}$ of the group* $[Q, L_q]$.

The multiplication table of this group is easily obtained. The element described in (5) is uniquely defined by y on Q $(y_n\neq 0)$. Let us call it T_y. If T_z is another element defined by z on Q $(z_n\neq 0)$, the product $T_yT_z=T_t$ is defined by t on Q $(t_n\neq 0)$. To obtain this product we need to get only the last column of coefficients. This yields

(6) $$T_yT_z=T_t$$
$$t_0=y_0+y_nz_0,\ t_1=y_1+y_nz_1,\cdots,t_{n-2}=y_{n-2}+y_nz_{n-2},\ t_{n-1}=\kappa,\ t_n=y_nz_n,$$

where the explicit form of κ is not material, it being uniquely determined by the fact that t is on Q.

We see from (6) that if y_n and z_n are 1, t_n is 1, and $T_yT_z=T_zT_y=T_t$. Also, if T_y, T_z are inverse, $y_nz_n=1$, or $z_n=1/y_n$. Hence, if $y_n=1$, $T_z^{-1}T_yT_z$ has a coefficient $\gamma_{n,n}=1/z_n\cdot 1\cdot z_n=1$. Thus

(7) *The elements of $g_{(s-1)s^{n-1}}$ in* (5) *for which $y_n=1$ form an abelian subgroup $g_{s^{n-1}}$ of order s^{n-1} [for s^{n-1} choices of $y_0,\cdots,y_{n-2}$] invariant under $g_{(s-1)s^{n-1}}$. This abelian subgroup is of type* $(1,1,\cdots,1)$, *contains only transformations of period p, and is a regular group on the s^{n-1} secants of Q on q.*

For, any line on a_{n-1} not in $x_n=0$, i. e., any secant, can be *represented* by the point where it cuts $x_{n-1}=0$, i. e., by $s_0,\cdots,s_{n-2},0,1$. This point is transformed by the element (5) into another point whose join with a_{n-1} is *represented* by

(8) $$(s_0+y_0)/y_n,\ (s_1+y_1)/y_n,\cdots,(s_{n-2}+y_{n-2})/y_n,\ 0,\ 1.$$

Thus, when $y_n=1$, we have

(9) $$s'_0=s_0+y_0,\cdots,s'_{n-2}=s_{n-2}+y_{n-2}.$$

Hence, given s and s', there is one and only one set of values $y_0,\cdots,y_{n-2},y_n=1$ for which (9) holds.

According to (6), if $y_n=\epsilon$, a primitive root in G. F., T_y is an element whose $(s-1)$-th power is in the above abelian group, whence

(10) *The factor group of* $g_{s^{n-1}}$ *with respect to* $g_{(s-1)s^{n-1}}$ *is a cyclic* g_{s-1}.

From this there follows that

(11) *The factors of composition of the collineation group of order* $(s-1)s^{n-1}N(n-2) = [Q, L_q]$ *in* (1) I 1, II 1, III 1 *are first the factors of composition of* Q_{n-2} [*cf.* **2** (10), (13), (16)], *second the factors of a cyclic* g_{s-1}, *and third the factors of the abelian* $g_{s^{n-1}}$ *of type* $(1, 1, \cdots, 1)$.

It is clear from the form of (5) that

(12) *Each of the* $s^{n-1}-1$ *elements of* $g_{s^{n-1}}$ *other than the identity has an* S_{n-2} *of fixed points,* $x_n = 0$ *and* $(y\partial/\partial x)Q(n-2)(x) = 0$ [cf. (4)]. *Each of these* S_{n-2}*'s in the* S_{n-1}, $x_n = 0$, *on* a_{n-1} $[P_{n-2} = (s^{n-1}-1)/(s-1)$ *in number*] *arises from* $s-1$ *transformations due to the factor in* $y_0, \cdots, y_{n-2}$.

If in (8) we set $s'_i = (s_i + y_i)/y_n$ $(i = 0, \cdots, n-2)$, then there is a fixed secant when $y_n \neq 1$, namely: $s_i = y_i/(y_n - 1)$. Hence

(13) *Any element in* $g_{(s-1)s^{n-1}}$ *not in* $g_{s^{n-1}}$ *has one and only one fixed secant on* $q = a_{n-1}$. *The subgroup of order* $s-1$ *which leaves one secant fixed is generated by pairs of involutions* I_p *for points* p *on the secant but not on* Q.

For, if we take $a_{n-1}a_n$ as a typical secant, this subgroup has the form $x'_i = x_i$ $(i = 0, \cdots, n-2)$, $x'_{n-1} = x_{n-1}/y_n$, $x'_n = y_n x_n$. This is the product of the pair of involutions, $x'_{n-1} = x''_n$, $x'_n = x''_{n-1}$; $x''_{n-1} = y_n x_n$, $x''_n = x_{n-1}/y_n$, the other variables being unaltered. For $y_n = -1$, this product is an involution of rank $(1, n-2)$, i. e., with a line and an S_{n-2} of fixed points. On the secant this effects the identity so that the g_{s-1} in (13) is, on the secant, the $g_{(s-1)/2}$ formed by products of an even number of reflections in real pairs apolar to a real pair.

The elements of $g_{(s-1)s^{n-1}}$ in $g_{s^{n-1}}$ may be obtained by taking a product of two products of pairs of involutions for one of which we have y_n, and for the other $z_n = 1/y_n$ [*cf.* (6)]. Hence

(14) *The group* $g_{(s-1)s^{n-1}}$ *is generated by products* $I_p I_{p'}$ *where* p, p' *are any two points not on* Q *but on a secant through* q.

With respect to the problem outlined in the introduction however, we want the group which leaves Q, L absolutely unaltered, and which is generated by involutions I_p for points p *on* the linear space L. Thus, our linear space being $x_n = 0$, this requirement restricts the transformations (5) for which

$x'_n = y_n x_n$ to those for which $y_n = 1$, i. e., to the subgroup $g_{s^{n-1}}$ of $g_{(s-1)s^{n-1}}$. On $q = a_{n-1}$ the tangents τ and generators γ of Q are cut by any S_{n-1} not on a_{n-1} in the points respectively, not on, and on, the quadric $Q(n-2)$. The points p on L but not on Q are points p_τ on these tangents. The involutions, $I_{p\tau}, I_{p'\tau}$ (p, p' on the same tangent τ), effect the same permutation on tangents τ and generators γ, when $I_{p\tau}I_{p'\tau}$ leaves each unaltered, and also leaves Q, L unaltered, and thus is in $g_{s^{n-1}}$. Conversely $g_{s^{n-1}}$ is generated by such pairs. To prove this let $z = z_0, \cdots, z_{n-2}, 0, 0$. If then $(\beta z)^2 \neq 0$, the point $p_\tau = z + \mu a_{n-1}$ $(\mu \neq 0)$ is a point on the tangent τ. The involution $I_{p\tau}$ is [cf. **2** (4)]

$$x' = x - (z + \mu a_{n-1})[2(\beta z)(\beta x) + \mu x_n]/(\beta z)^2.$$

If $I_{p'\tau}$ is another point on the same tangent determined by $\mu' \neq \mu$, then

$$\begin{aligned} I_{p\tau}I_{p'\tau} : x' = x &+ z \cdot (\mu - \mu')x_n/(\beta z)^2 \\ &+ a_{n-1} \cdot (\mu' - \mu)[2(\beta z)(\beta x) + (\mu - \mu')x_n]/(\beta z)^2. \end{aligned}$$

If we compare this with (5) we find that

$$\begin{gathered} -(\mu - \mu')^2/(\beta z)^2, 1, \\ y_0, \cdots, y_n = (\mu - \mu')z_0/(\beta z)^2, \cdots, (\mu - \mu')z_{n-2}/(\beta z)^2, \end{gathered}$$

and thus $I_{p\tau}I_{p'\tau}$ is in $g_{s^{n-1}}$ since $y_n = 1$. We do not however find in this way all the elements of $g_{s^{n-1}}$ due to the restriction, $(\beta z)^2 \neq 0$. If then we choose z, z' not on $Q(n-2)$ such that $z + z' = t$ is on $Q(n-2)$, the element of $g_{s^{n-1}}$ corresponding to $y = t$ is $I_{p\tau}I_{p'\tau}I_{q\tau'}I_{q'\tau'}$ where τ' is the tangent determined by z'. Hence

(15) *The group $g_{s^{n-1}}$ is generated by products of pairs of involutions I_p for pairs of points p on tangents τ of Q in L_q.*

The involution $I_{p\tau}$ effects the same permutation on lines λ on a_{n-1} in L as the involution I_p for $Q(n-2)$ effects on the points of S_{n-2}. Hence these involutions generate the $G(n-2)(I_p)$ on the lines λ and in S_n the $G(I_{p\tau})$ has the invariant $g_{s^{n-1}}$ with factor group $G(n-2)(I_p)$. Hence

(16) *The collineation group, $[G(I_p), L_q]$, generated by involutions I_p for points p on L_q, and thus leaving Q, L_q unaltered, has factors of composition which are the factors of $G(n-2)(I_p)$ [cf. **2** (10), (13), (16)], and the factors of $g_{s^{n-1}}$.*

We now consider the cases (1) $I(2)I(3)$. Given $Q(n)$ and an outside

point o, the polar space L_o cuts $Q(n)$ in a $Q_+(n-1)$. The order of the entire collineation group of $Q(n)$, L_o is $N_+(n-1)$. This we write as $2 \cdot N_+(n-1)/2$, since we are interested only in collineations which leave $Q(n)$, L_o, and therefore the section $Q_+(n-1)$, absolutely unaltered. The latter group is generated by involutions I_p for points p on L_o. But the involutions attached to n linearly independent points in L_o generate the involution I_o which necessarily is invariant under the entire group. Similar remarks apply to an inside point i and the section $Q_-(n-1)$. Hence

(17) *The collineation group,* $[G(n)(I_p), L_o]$ $\{[G(n)(I_p), L_i]\}$ *generated by involutions* I_p *for points* p *on* $L_o\{L_i\}$ *has for factors of composition those of the group* $G_+(n-1)(I_p)$ $\{G_-(n-1)(I_p)\}$, *and a factor corresponding to the invariant* g_2 *generated by* $I_o\{I_i\}$.

There remain finally the cases II 2, III 2. We pass immediately to the $G(I_p)$ and, in connection with (2), state that:

(18) *The collineation group* $[G_\pm(n)(I_p), L_{o,i}]$ *generated by involutions* I_p *for points* p *on* L_o *or* L_i, *has for factors of composition those of the group* $G(n-1)(I_p)$ *and a factor* 2 *corresponding to the invariant* g_2 *generated by* I_o *or* I_i.

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ON THE DISTRIBUTION OF THE VALUES OF THE RIEMANN ZETA FUNCTION.

By H. Bohr and B. Jessen.

Introduction. The object of this note is to fill in a gap in the description of the distribution of the values of the Riemann zeta function $\zeta(s) = \zeta(\sigma + it)$, or rather the function $\log \zeta(s)$, in the half-plane $\sigma > 1$.[1] From the Euler product we have for this function the expression

$$-\log \zeta(s) = \sum_{n=1}^{\infty} \log (1 - p_n^{-s}) = \sum_{n=1}^{\infty} \log (1 - p_n^{-\sigma} e^{-it \log p_n}),$$

where p_n denotes the prime numbers $2, 3, 5, \cdots$.[2]

For a fixed $\sigma > 1$ we consider in the complex w-plane the closure $M(\sigma)$ of the set of values $-\log \zeta(\sigma + it)$, $-\infty < t < +\infty$. It is known that, on account of the linear independence of the numbers $\log p_n$, this set $M(\sigma)$ is identical with the range of values of the function

$$F(\theta_1, \theta_2, \cdots) = \sum_{n=1}^{\infty} \log (1 - p_n^{-\sigma} e^{i\theta_n}),$$

where the real variables $\theta_1, \theta_2, \cdots$ are independent of each other and θ_n describes the interval $0 \leqq \theta_n < 2\pi$. Thus, if for an arbitrary r in $0 < r < 1$ we denote by $S(r)$ the curve

(1) $$w = \log (1 + x), \qquad |x| = r,$$

we have

(2) $$M(\sigma) = \sum_{n=1}^{\infty} S(p_n^{-\sigma}),$$

where the sum of sets is to be taken in the vectorial sense, that is, the sum denotes the set of all points $w = \sum_{n=1}^{\infty} w_n$, where w_n belongs to $S(p_n^{-\sigma})$.

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[1] The results concerning the distribution of the values of the function $\log \zeta(s)$ in the half-plane $\sigma > 1$ which we shall use and which are restated in the text are given in H. Bohr [2]. For a more detailed study of the distribution of the values involving also problems of probability and dealing with the half-plane $\sigma > \frac{1}{2}$, we refer to H. Bohr and B. Jessen [5-6], and, particularly, to the comprehensive treatment in B. Jessen and A. Wintner [9]. The results concerning the function $\zeta'(s)/\zeta(s)$ which we mention are given in H. Bohr [1] and C. Burrau [7].

[2] We consider $-\log \zeta(s)$ instead of as usual $\log \zeta(s)$ itself in order to avoid the minus sign in several other places. We notice that all occurring infinite series are absolutely convergent.

From this representation of $M(\sigma)$ and the simple fact, to which we return below, that each curve $S(r)$ is convex, has been obtained the following simple result concerning the shape of the set $M(\sigma)$: that $M(\sigma)$ is for each $\sigma > 1$ either a closed domain bounded by a single convex curve $A(\sigma)$ or a closed ring-shaped domain, bounded by two convex curves $A(\sigma)$ and $B(\sigma)$, where $B(\sigma)$ lies inside $A(\sigma)$. Furthermore, it was shown by rough estimations that for all σ sufficiently near to 1 the first case occurs, while the second case occurs for all sufficiently large σ.[3]

So far the situation is quite similar to that obtained for the derivative $\zeta'(s)/\zeta(s)$ of the function $\log \zeta(s)$, only in this latter case the situation is simplified by the fact that the convex curves to be added turn out to be circles. Their sum is therefore either the closed surface of a circle or a closed concentric circular ring. In this case it was possible by simple computations to decide for which σ each of the two cases occurred, the result being the existence of a constant $D > 1$, such that for $\sigma \leqq D$ the case of the circle, for $\sigma > D$ the case of the circular ring takes place. A numerical calculation gave the approximate value $D = 2.576076$.

The object of the present note is to prove that a quite analogous situation holds for the function $\log \zeta(s)$ itself.

THEOREM. *Denoting by $M(\sigma)$ for $\sigma > 1$ the closure of the set of values $-\log \zeta(\sigma + it)$, $-\infty < t < +\infty$, there exists a constant $C > 1$, so that for each $\sigma \leqq C$ the set $M(\sigma)$ is a closed domain bounded by a single convex curve $A(\sigma)$, while for each $\sigma > C$ the set $M(\sigma)$ is a closed ring-shaped domain bounded by two convex curves $A(\sigma)$ and $B(\sigma)$, where $B(\sigma)$ lies inside $A(\sigma)$.*

C is characterized as the only root in $\sigma > 1$ of the equation

$$\arcsin 2^{-\sigma} = \sum_{n=2}^{\infty} \arcsin p_n^{-\sigma}.$$

We have the approximate value $C = 1.764$, correct to two decimal places.

Some details regarding the shape of the curve $B(\sigma)$ for $\sigma > C$ are contained in a theorem at the end of this note.

A geometrical criterion that the set be ring-shaped. In our investigations of the set $M(\sigma)$ as given by the formula (2) we shall make no use of any special properties of the prime numbers p_n except that $1 < p_1 < p_2 < \cdots$

[3] For the problem of the addition of convex curves see H. Bohr [3], H. Bohr and B. Jessen [4] and E. K. Haviland [8]. A short account is to be found in B. Jessen and A. Wintner [9].

and that the series $\sum_{n=1}^{\infty} p_n^{-\sigma}$ converges for $\sigma > 1$ and diverges for $\sigma = 1$. It will therefore be more natural to consider the general case, where the sets $M(\sigma)$ for $\sigma > 1$ are defined by

$$M(\sigma) = \sum_{n=1}^{\infty} S(e^{-\lambda_n \sigma}), \tag{3}$$

where $0 < \lambda_1 < \lambda_2 < \cdots$ is any sequence such that the series $\sum_{n=1}^{\infty} e^{-\lambda_n \sigma}$ converges for $\sigma > 1$ and diverges for $\sigma = 1$.[4]

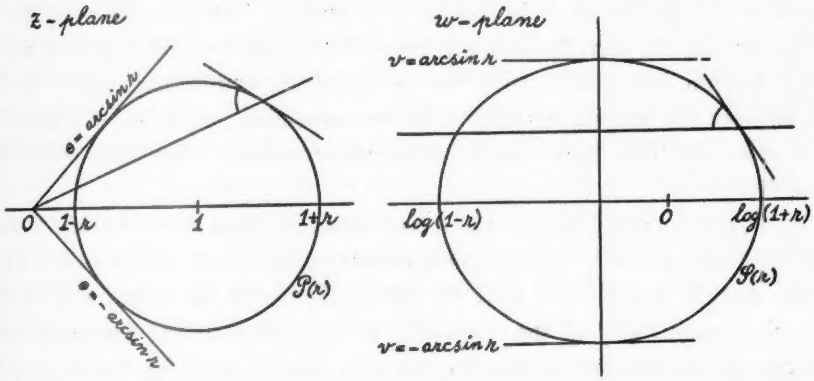

FIGURE 1.

We begin with some simple remarks concerning the curve $S(r)$, $0 < r < 1$, given by the representation (1). When x describes the circle $|x| = r$, the point $z = 1 + x$ describes the circle $P(r)$ in the z-plane with mid-point 1 and radius r (see Figure 1). The curve $S(r)$ is the image of this circle $P(r)$ by the conformal representation $w = \log z$. Writing $z = \rho e^{i\theta}$, $-\pi/2 < \theta < \pi/2$, and $w = u + iv$, we have

$$u = \log \rho \quad \text{and} \quad v = \theta.$$

The ranges of ρ and θ, when z describes $P(r)$, are

$$1 - r \leqq \rho \leqq 1 + r \quad \text{and} \quad -\operatorname{arc\,sin} r \leqq \theta \leqq \operatorname{arc\,sin} r.$$

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[4] Evidently the constant C occurring in the theorem will depend on the sequence $\lambda_1, \lambda_2, \cdots$. We notice that in the special case, where the λ_n are linearly independent, the set $M(\sigma)$ given by (3) is the closure of the set of values $f(\sigma + it)$, $-\infty < t < +\infty$, where $f(s)$ is the analytic function

$$f(s) = \sum_{n=1}^{\infty} \log(1 - e^{-\lambda_n s}).$$

Thus our theorem holds for any function of this kind.

The extreme values of ρ are taken at the points of $P(r)$ lying on the real axis and correspond to the value $\theta=0$; the extreme values of θ are taken at the touching points of the tangents from the origin to $P(r)$ and correspond to the value $\rho=(1-r^2)^{\frac{1}{2}}$. To any value of ρ except the extreme ones correspond two values of θ, which are numerically equal but with opposite signs, while to each value of θ except the extreme ones correspond two values of ρ, whose product is $1-r^2$. Consequently, the curve $S(r)$ is symmetrical with respect to the lines $v=0$ and $u=\frac{1}{2}\log(1-r^2)$ and is cut by each line $u=u_0$, $\log(1-r)<u_0<\log(1+r)$, and by each line $v=v_0$, $-\arcsin r<v_0<\arcsin r$, in exactly two points. Furthermore, the curve $S(r)$ is convex; in fact, the angle between the tangent of $S(r)$ at the point (u_0, v_0) and the line $v=v_0$ is, in the virtue of the conformity, equal to the angle between the tangent of $P(r)$ at the corresponding point (ρ_0, θ_0) and the line $\theta=\theta_0$, and this latter angle varies monotonously when the point describes $P(r)$.[5]

By $S_0(r)$ we shall denote the curve obtained from $S(r)$ by the translation $-\frac{1}{2}\log(1-r^2)$, having the symmetry axes $v=0$ and $u=0$. It is obvious that, if $0<r''<r'<1$, the image $S(r'')$ of the circle $P(r'')$ lies inside the image $S(r')$ of the circle $P(r')$. In virtue of the symmetry and convexity of the two curves this implies that $S_0(r'')$ must lie inside $S_0(r')$.

The symmetry of the single curve $S(r)$ immediately involves a similar symmetry of the set $M(\sigma)$ defined by (3), the axes of symmetry being $v=0$ and $u=\sum_{n=1}^{\infty}\frac{1}{2}\log(1-e^{-2\lambda_n\sigma})$. By $M_0(\sigma)$ we shall denote the set obtained from $M(\sigma)$ by the translation $-\sum_{n=1}^{\infty}\frac{1}{2}\log(1-e^{-2\lambda_n\sigma})$, having the symmetry axes $v=0$ and $u=0$. Obviously

$$M_0(\sigma)=\sum_{n=1}^{\infty}S_0(e^{-\lambda_n\sigma}).$$

We shall now deduce a simple criterion enabling us to decide whether for a given σ the set $M_0(\sigma)$ and hence also the set $M(\sigma)$ is ring-shaped or convex. For this purpose we write $M_0(\sigma)$ in the form

$$M_0(\sigma)=S_0(e^{-\lambda_1\sigma})+N_0(\sigma),\quad \text{where}\quad N_0(\sigma)=\sum_{n=2}^{\infty}S_0(e^{-\lambda_n\sigma}).$$

[5] The two half axes of $S(r)$ lying on the two axes of symmetry $v=0$ and $u=\frac{1}{2}\log(1-r^2)$ are

$$a(r)=\tfrac{1}{2}[\log(1+r)-\log(1-r)] \quad\text{and}\quad b(r)=\arcsin r$$

respectively. For the orientation of the reader, we notice without proof that the ratio $a(r)/b(r)$ is an increasing function of r in $0<r<1$, so that the curve $S(r)$, which for small values of r is approximately a circle, becomes more and more oblong as r increases.

As the set $M_0(\sigma)$ has the origin as center of symmetry, it is clear that it is convex or ring-shaped according as it contains or does not contain the origin, that is, according as there exist or do not exist points w_1 and w_2 of $S_0(e^{-\lambda_1\sigma})$ and $N_0(\sigma)$ respectively, such that $w_1 + w_2 = 0$, which by the symmetry of either set with respect to the origin is the case according as $S_0(e^{-\lambda_1\sigma})$ and $N_0(\sigma)$ have or have not points in common.

Now as a sum of convex curves the set $N_0(\sigma)$ is itself either a closed domain bounded by a simple convex curve $C_0(\sigma)$ or a closed ring-shaped domain bounded by two convex curves $C_0(\sigma)$ and $D_0(\sigma)$, where $D_0(\sigma)$ lies inside $C_0(\sigma)$. From the general considerations regarding the addition of convex curves and the fact that all the curves $S_0(e^{-\lambda_n\sigma})$, $n \geqq 3$, surround the origin and are contained in $S_0(e^{-\lambda_2\sigma})$, it follows immediately that the interior boundary $D_0(\sigma)$ of $N_0(\sigma)$, if it exists, must lie inside $S_0(e^{-\lambda_2\sigma})$ and hence still more inside $S_0(e^{-\lambda_1\sigma})$. Thus in the determination, whether $S_0(e^{-\lambda_1\sigma})$ and $N_0(\sigma)$ have or have not points in common, it will make no difference if, in the case where $D_0(\sigma)$ exists, we add to $N_0(\sigma)$ the interior of $D_0(\sigma)$, so that in all cases the problem is only to decide whether $S_0(e^{-\lambda_1\sigma})$ and the closed convex domain bounded by the curve $C_0(\sigma)$ have or have not points in common. Hence we have the following criterion:

The set $M_0(\sigma)$ is ring-shaped or convex according as the convex curve $C_0(\sigma)$ lies or does not lie inside $S_0(e^{-\lambda_1\sigma})$.

An analytical formulation of the criterion. That a convex curve lies inside another convex curve may be expressed analytically in various ways. For the present purpose, where one of the two curves to be considered is given as the exterior boundary for a sum of convex curves, the obvious procedure is to use the supporting functions (Stützfunktion) of the two curves, since the supporting function of the exterior boundary of a sum of convex curves is immediately expressed as the sum of the supporting functions for the single curves to be added.

For each r in $0 < r < 1$ we denote by $H_0(r;\alpha)$, where α is an angular variable, the supporting function of the convex curve $S_0(r)$, defined as the maximum of $u\cos\alpha + v\sin\alpha$ when (u,v) describes $S_0(r)$. The explicit expression for $H_0(r;\alpha)$ is somewhat complicated, but will not be needed; we shall use only the special value $H_0(r;\pi/2) = \arcsin r$. Denoting by $K_0(\sigma;\alpha)$ the supporting function of the curve $C_0(\sigma)$, we have

$$K_0(\sigma;\alpha) = \sum_{n=2}^{\infty} H_0(e^{-\lambda_n\sigma};\alpha).$$

Now if S' and S'' are two convex curves and $H'(\alpha)$ and $H''(\alpha)$ their supporting functions, the necessary and sufficient condition that S'' lie inside S' is that $H'(\alpha) > H''(\alpha)$ for all α. We may therefore state the above criterion in the following form:

The set $M(\sigma)$ is ring-shaped or convex according as the inequality

$$H_0(e^{-\lambda_1\sigma};\alpha) > \sum_{n=2}^{\infty} H_0(e^{-\lambda_n\sigma};\alpha) \tag{4}$$

holds or does not hold for all α. For reasons of symmetry it is evidently sufficient to consider values $0 \leqq \alpha \leqq \pi/2$.

Proof of the theorem. By virtue of the last criterion our theorem will follow from the following two propositions:

(i) *For any fixed $\sigma > 1$ the inequality* (4) *will hold for all α, if it holds for $\alpha = \pi/2$.*

(ii) *There exists a constant $C > 1$ such that the inequality*

$$H_0(e^{-\lambda_1\sigma};\pi/2) > \sum_{n=2}^{\infty} H_0(e^{-\lambda_n\sigma};\pi/2)$$

is or is not satisfied, according as $\sigma > C$ or $\sigma \leqq C$.

Proof of proposition (i). For a fixed $\sigma > 1$ we put $e^{-\lambda_n\sigma} = r_n$, so that $r_1 > r_2 > \cdots$. The proposition to be proved is then that the inequality

$$H_0(r_1;\alpha) > \sum_{n=2}^{\infty} H_0(r_n;\alpha)$$

holds for $0 \leqq \alpha < \pi/2$, if it holds for $\alpha = \pi/2$. Obviously this will be proved if we prove that for $0 < r'' < r' < 1$ the inequality

$$\frac{H_0(r'';\alpha)}{H_0(r'';\pi/2)} < \frac{H_0(r';\alpha)}{H_0(r';\pi/2)},$$

that is, the inequality

$$\frac{H_0(r'';\alpha)}{\arcsin r''} < \frac{H_0(r';\alpha)}{\arcsin r'},$$

holds for $0 \leqq \alpha < \pi/2$.

Denoting for an arbitrary r in $0 < r < 1$ by $S^*_0(r)$ the convex curve similar to $S_0(r)$ with respect to the origin in the ratio $1/(\arcsin r)$, the supporting function $H^*_0(r;\alpha)$ of $S^*_0(r)$ is

$$H^*_0(r;\alpha) = \frac{H_0(r;\alpha)}{\arcsin r},$$

so that the assertion to be proved is that for $0 < r'' < r' < 1$ the inequality

$$H^*_0(r''; \alpha) < H^*_0(r'; \alpha)$$

holds for $0 \leqq \alpha < \pi/2$. This inequality simply expresses that the curve $S^*_0(r'')$ lies inside $S^*_0(r')$ except for the two points $(0, 1)$ and $(0, -1)$, which lie on all the curves $S^*_0(r)$.[6]

For an arbitrary r in $0 < r < 1$ we represent the part of the curve $S^*_0(r)$ lying in $u \geqq 0$, $v \geqq 0$ by an equation $u = f(r; v)$, $0 \leqq v \leqq 1$. Denoting for a fixed t in $0 < t < 1$ by $\alpha(r; t)$ the angle between the line $v = t$ and the normal of the curve $S^*_0(r)$ at the point $(f(r; t), t)$, we have $\tan \alpha(r; t) = -f'_v(r; t)$, so that

$$f(r; v) = \int_v^1 \tan \alpha(r; t)\, dt$$

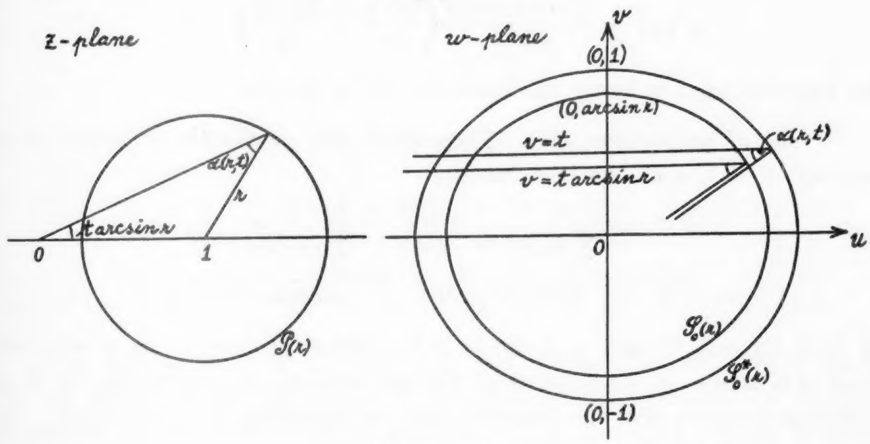

FIGURE 2.

for $0 \leqq v < 1$. Our assertion being that for $0 < r'' < r' < 1$ we have $f(r''; v) < f(r'; v)$ in $0 \leqq v < 1$, it is therefore sufficient to prove that for any fixed t in $0 < t < 1$ the angle $\alpha(r; t)$ is an increasing function of r in $0 < r < 1$.

In order to calculate the angle $\alpha(r; t)$ we notice that $\alpha(r; t)$ is also the angle between the line $v = t \arcsin r$ and the normal of the curve $S_0(r)$ or $S(r)$ at one of the points lying on this line, and hence by the conformal representation equal to the angle between the line $\theta = t \arcsin r$ and the normal of the circle $P(r)$ at one of the points lying on this latter line (see Figure 2). Hence we have the relation

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[6] This contains the statement in the preceding footnote, that the curve $S(r)$ becomes more and more oblong as r increases, and shows that this even happens in a very regular way.

$$\frac{\sin \alpha(r;t)}{1} = \frac{\sin (t \arcsin r)}{r}.$$

Introducing $\arcsin r = y$ as new variable instead of r, we find

$$\sin \alpha(r;t) = \frac{\sin ty}{\sin y}.$$

Thus in order to prove that for any fixed t in $0 < t < 1$ the angle $\alpha(r;t)$ is an increasing function of r in $0 < r < 1$, we have to prove only that

$$g(y) = \frac{\sin ty}{\sin y}$$

is an increasing function of y in $0 < y < \pi/2$, which is clear since

$$g'(y) = \frac{ty \cos y \cos ty}{\sin^2 y}\left(\frac{\tan y}{y} - \frac{\tan ty}{ty}\right) > 0,$$

the function $\tan x/x$ being increasing in $0 < x < \pi/2$.

Proof of proposition (ii). Proposition (ii) states the existence of a constant $C > 1$, such that the function

$$f(\sigma) = \frac{\sum_{n=2}^{\infty} H_0(e^{-\lambda_n \sigma}; \pi/2)}{H_0(e^{-\lambda_1 \sigma}; \pi/2)} = \frac{\sum_{n=2}^{\infty} \arcsin e^{-\lambda_n \sigma}}{\arcsin e^{-\lambda_1 \sigma}}$$

is $\geqq 1$ for $\sigma \leqq C$ and < 1 for $\sigma > C$. Since $f(\sigma) \to \infty$ as $\sigma \to 1$ and $f(\sigma) \to 0$ as $\sigma \to \infty$ on account of the assumptions concerning the λ_n, it is sufficient to prove that the function $f(\sigma)$ is decreasing in $1 < \sigma < \infty$. We shall even prove that each term

$$g(\sigma) = \frac{\arcsin e^{-\lambda_n \sigma}}{\arcsin e^{-\lambda_1 \sigma}}$$

has a negative derivative $g'(\sigma)$ and hence is decreasing in $1 < \sigma < \infty$. The logarithmic derivative $g'(\sigma)/g(\sigma)$ of the function $g(\sigma)$ being the difference between the logarithmic derivatives of the numerator and the denominator, the inequality $g'(\sigma) < 0$ will be proved if we prove that the logarithmic derivative $h'(\sigma)/h(\sigma)$ of the function

$$h(\sigma) = \arcsin e^{-\lambda \sigma}$$

is for each fixed σ in $1 < \sigma < \infty$ a decreasing function of λ in $0 < \lambda < \infty$. Now

$$\frac{h'(\sigma)}{h(\sigma)} = \frac{-\lambda e^{-\lambda \sigma}}{\sqrt{1 - e^{-2\lambda \sigma}} \arcsin e^{-\lambda \sigma}}.$$

For a fixed σ we introduce arc sin $e^{-\lambda\sigma} = y$ as new variable instead of λ, and have then to prove that the function

$$\phi(y) = \frac{1}{\sigma}\,\frac{\sin y \log \sin y}{\cos y \cdot y} = \frac{1}{\sigma}\,\frac{\tan y \log \sin y}{y} = \frac{1}{\sigma}\,\frac{\psi(y)}{y}$$

is increasing in the interval $0 < y < \pi/2$. Since $\psi(0) = 0$, this will certainly be the case if $\psi(y)$ is convex in $0 < y < \pi/2$, so that it is sufficient to prove that

$$\psi'(y) = 1 + \frac{\log \sin y}{\cos^2 y}$$

is increasing in $0 < y < \pi/2$. Once more changing the variable, putting $\sin y = t$, we have thus only to show that

$$\chi(t) = \frac{\log t}{1 - t^2}$$

is increasing in $0 < t < 1$. But this is clear since

$$\chi'(t) = \frac{1 - t^2 + 2t^2 \log t}{t(1-t^2)^2} = \frac{\xi(t)}{t(1-t^2)^2} > 0$$

in $0 < t < 1$, as $\xi'(t) = 4t \log t < 0$ in $0 < t < 1$ and $\xi(1) = 0$.

Another theorem. It is easily seen that the exterior boundary $A(\sigma)$ of the set $M(\sigma)$ for an arbitrary $\sigma > 1$ contains neither corners nor straight segments. By arguments similar to those applied above, we are able to prove the following theorem regarding the shape of the interior boundary $B(\sigma)$ of the set $M(\sigma)$ for $\sigma > C$:

THEOREM. *There exists a constant $E > C$ such that for each $\sigma < E$ the curve $B(\sigma)$ has exactly two corners lying on the real axis, while for each $\sigma \geqq E$ the curve $B(\sigma)$ has no corners. For no value of σ does $B(\sigma)$ contain straight segments.*

E is characterized as the only root in $\sigma > 1$ of the equation

$$2^{-\sigma} = \sum_{n=2}^{\infty} p_n^{-\sigma}.$$

We have the approximate value $E = 1.778$, correct to two decimal places.

We are indebted to Mr. J. P. Møller for the numerical calculation of the constants C and E.

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ON A CLASS OF FOURIER TRANSFORMS.

By Aurel Wintner.

The present paper deals with the Fourier analysis of certain analytic functions. § 1 collects the tools to be used. Theorems I and II of § 2 concern the Fourier analysis of a class of meromorphic functions which are reciprocal values of even transcendental entire functions $f(z)$ of genus 0 in z^2. This theorem is suggested by Hausdorff's remarks on the function $1/\operatorname{Cosh} t$ (cf. Hausdorff [8]). Theorem VI furnishes the general analytical background to the well known Fourier representation of $|\Gamma(b+it)|^2$, where $b>0$, in the same manner as Theorem I yields the general background of the standard Fourier representation of $1/\operatorname{Cosh} t$. Theorem VII is another analogue of Theorem I and concerns the case of an " erhöht " genus 0 in the sense of Pólya (cf. [19], where further references are given). While Theorem I leads, as shown by Theorem III, to a strange consequence of Riemann's hypothesis, Theorem IV is independent of this hypothesis. Theorem V deals with an interesting transcendant defined by a Bernoulli convolution. The elementary Theorem X and the remarks which follow it treat distributions derived by projection from equidistributions which belong to the interior or to the boundary of an n-dimensional sphere. Theorems VIII and IX contain new information about a class of transcendants introduced by Cauchy [4] which play an important rôle in the investigations of Lévy and Pólya on the foundations of the analytic theory of probability; the same transcendants also occur in the Hardy-Littlewood treatment of Waring's problem (for references cf. Pólya [18]). It is shown by Theorems XX and XXI that Theorems VIII and IX may be extended from the case of trigonometrical integrals to the case of integrals containing Bessel functions. As an application of Theorem VIII, it is shown in § 3 that the three standard postulates in the theory of error distribution, which go back to Gauss [6], are not independent of each other, one of them being implied by the two others. As shown by Theorem XIX, a corresponding result holds in the multidimensional case also. The other results proven at the end of § 4 are known in the case $n=1$, where n is the dimension number (cf. Lévy [14], Pólya [17], Wintner [23]). The difference between the cases $n=1$ and $n>1$ is about the same as that between an ordinary and a partial differential equation. It turns out, however, that the arbitrary function contained in the solution of the problem in the case $n>1$ must be a constant due to the " boundary condition " of a finite

dispersion. This result might have some physical interest in view of the Maxwell assumption on velocity distribution at random ($n = 3$).

INDEX.

1. *The Fourier-Stieltjes transform.* Let $\sigma = \sigma(x)$, $-\infty < x < +\infty$, be a distribution function, i. e., a monotone function for which $\sigma(-\infty) = 0$ and $\sigma(+\infty) = 1$. It may be assumed that

$$\sigma(x) = \tfrac{1}{2}\{\sigma(x+0) + \sigma(x-0)\}$$

holds also when x belongs to the sequence of discontinuity points (if any). If c is a positive constant, $\sigma(cx)$ also is a distribution function; it will be termed similar to $\sigma(x)$. The function $1 - \sigma(-x)$ also is a distribution function and may be called the conjugate of $\sigma(x)$. The conjugate of the conjugate of σ is σ itself. σ will be said to be symmetric if it is identical with its conjugate, i. e., if

$$\sigma(x) + \sigma(-x) \equiv 1.$$

If σ is symmetric and if the curve $\sigma = \sigma(x)$ is concave (from below) in the open interval $0 < x < +\infty$, hence convex in the open interval $-\infty < x < 0$, then σ will be termed a convex distribution function. It is well known that if a function is convex and bounded in an open interval, it is absolutely continuous in this interval. Thus a convex distribution function $\sigma(x)$ is absolutely continuous in every interval not containing $x = 0$. That a convex distribution function may be discontinuous at $x = 0$, is shown by the example $\chi(x) = \frac{1}{2}(1 + \operatorname{sg} x)$, where $\operatorname{sg} x = -1$, 0 or 1 according as $x < 0$, $= 0$ or > 0. This distribution function χ plays the rôle of the unit in what may be called the algebra of distribution functions.

A sequence $\{\sigma_n\}$ of distribution functions is said to be convergent if there exists a distribution function σ such that $\sigma_n(x) \to \sigma(x)$ at every continuity point x of σ. This is what will be meant by writing $\sigma_n \to \sigma$. Thus $\sigma_n \to \sigma$ and $\sigma_n \to \rho$ imply $\sigma = \rho$. It is clear that if every σ_n is symmetric, then so is σ and that if every σ_n is convex in the sense defined above, then so is σ.

The spectrum of a distribution function σ is defined as the set of those

points $x = x_0$ for which $\sigma(x') < \sigma(x'')$ whenever $x' < x_0 < x''$. This terminology is in accordance with the usual physical terminology concerning the frequency distribution determined by σ, i. e., with the Wirtinger-Hilbert terminology in the theory of linear differential and integral equations and is, therefore, at variance with a terminology recently proposed by Wiener [21], p. 163. The discontinuity points of σ, if any, clearly belong to the spectrum. If σ is analytic, then the spectrum is the whole x-axis. If there exists a finite or infinite sequence $\{x_n\}$ of distinct points such that

$$\sum_n [\sigma(x_n + 0) - \sigma(x_n - 0)] = \int_{-\infty}^{+\infty} d\sigma(x), \text{ i. e., } = 1,$$

then σ will be termed purely discontinuous. The spectrum may be the whole x-axis in this case also.

If two independent random variables ξ_1, ξ_2 have the distribution functions σ_1, σ_2, then the distribution function of $\xi_1 + \xi_2$ is denoted by $\sigma_1 * \sigma_2$ and is termed the convolution ("Faltung") of σ_1 and σ_2; it is represented at its continuity points x by the integral

$$\int_{-\infty}^{+\infty} \sigma_1(x - y)\, d\sigma_2(y)$$

(cf., e. g., Hausdorff [8]). It is easy to see that $\sigma_1 * \sigma_2 = \sigma_2 * \sigma_1$ and $(\sigma_1 * \sigma_2) * \sigma_3 = \sigma_1 * (\sigma_2 * \sigma_3)$. Also, $\sigma * \chi = \sigma$ for any σ, where again $\chi(x) = \frac{1}{2}(1 + \operatorname{sg} x)$. The conjugate of $\sigma_1 * \sigma_2$ is the convolution of the conjugates of σ_1 and of σ_2. It follows that if σ_1 and σ_2 are symmetric, then so is $\sigma_1 * \sigma_2$. Furthermore, if σ_1 and σ_2 are convex, then so is $\sigma_1 * \sigma_2$. The truth of the last statement is implied by the treatment of a problem on rearrangements (cf. Hardy-Littlewood-Pólya [7], pp. 273-274); a direct proof may be found in § 4, where the theorem is extended to the multi-dimensional case (Theorem XIII). The truth of the statement is almost trivial in view of the statistical interpretation of the convolution process, mentioned above (cf. § 3). Examples show that if $\sigma_1 * \sigma_2$ and σ_2 are convex, then σ_1 need not be convex; cf., e. g., the convex distribution function occurring in Theorem V.

The Fourier transform of a distribution function $\sigma(x)$ is defined as the Stieltjes integral

$$(1) \qquad L(t) = L(t; \sigma) = \int_{-\infty}^{+\infty} e^{itx} d\sigma(x), \text{ where } -\infty < t < +\infty.$$

It is easy to see that the function $L(t; \sigma)$ is uniformly continuous in the

infinite interval $-\infty < t < +\infty$; it may be nowhere absolutely continuous, since the Weierstrass example

$$(1-a)\sum_{k=0}^{\infty} a^k \exp(ib^k t)$$

is, for suitable values of a and b, an $L(t;\sigma)$ which is nowhere differentiable. There belongs to every $L(t;\sigma)$ but one $\sigma(x)$, since

$$\sigma(x) = \sigma(0) + (2\pi i)^{-1}\int_{-\infty}^{+\infty} t^{-1}(1-e^{-itx})L(t;\sigma)\,dt, \text{ where } \int_{-\infty}^{+\infty} = \lim_{T=+\infty}\int_{-T}^{T}.$$

This is Lévy's inversion formula (cf., e.g., Wiener [21], Theorem 36 or Haviland [10]). It implies that if the integral of $|L(t;\sigma)|$ over $-\infty < t < +\infty$ is finite, then $\sigma(x)$ has for $-\infty < x < +\infty$ a uniformly continuous and bounded derivative which may be obtained by formal differentiation,

$$\sigma'(x) = (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{-ixt}L(t;\sigma)\,dt. \tag{2}$$

If there exist two positive constants, a and λ, such that

$$L(t;\sigma) = O(\exp\{-a\,|\,t\,|^{\lambda}\}) \text{ as } t \to \pm\infty, \tag{3}$$

then all derivatives of $\sigma(x)$ exist for every x and may be obtained by successive formal differentiation of (2). If in particular $\lambda = 1$, then there exists in a strip $-\alpha < y < \alpha$ of the z-plane, where $z = x + iy$, a regular and bounded function which becomes the distribution function $\sigma(x)$ if $y = 0$, the least upper bound of the admissible values of the width 2α being not less than $2a$; cf. Wintner [25]. Hence if $\lambda > 1$, then one may choose 2α arbitrarily large, and so σ is an entire function which is bounded in every strip parallel to the real axis. It may be mentioned that σ cannot be a rational entire function, since $\sigma(-\infty) = 0$ and $\sigma(+\infty) = 1$.

It is obvious from (1) that $L(-t;\sigma) = L(t;\bar{\sigma})$, where $\bar{\sigma}(x) = 1-\sigma(-x)$. Furthermore, $L(t;\sigma)$ and $L(-t;\sigma)$ are conjugated complex values. Since $\sigma(x)$ is monotone non-decreasing and has the total variation 1, it is also clear from (1) that $L(0;\sigma) = 1$ and that $|L(t;\sigma)| \leqq 1$ for every t. Suppose that σ is such that $|L(t;\sigma)| = 1$ holds not only for $t = 0$ but for at least one $t = t_0 \neq 0$ also. Then $L(0;\sigma) - e^{i\vartheta}L(t_0;\sigma) = 0$ for some real $\vartheta = \vartheta(t_0)$; hence, on taking the real part,

$$\int_{-\infty}^{+\infty} f(x)\,d\sigma(x) = 0, \text{ where } f(x) = 1-\cos(\vartheta + t_0 x).$$

Consequently, since $f(x) \geqq 0$ is continuous, $f(x) = 0$ at every point x of the spectrum of σ. Now every root of the equation $f(x) = 0$ satisfies the congruence $t_0 x \equiv -\vartheta \pmod{2\pi}$, where $t_0 \neq 0$ and $\vartheta = \vartheta(t_0)$. Hence in order that $|L(t;\sigma)| = 1$ holds for at least one $t \neq 0$, it is necessary that the spectrum of σ be a sequence contained in an arithmetical progression. This condition (cf. Cannon and Wintner [2]) is sufficient also, since if it is satisfied, $L(t;\sigma)$ is a periodic function and must, therefore, attain the value attained at $t = 0$ not only at $t = 0$. If $|L(t;\sigma)| = 1$ for every t, i. e., if t_0 is arbitrary, then the spectrum must be contained in an arithmetical progression of arbitrary difference, hence it must consist of a single point $x = b$. The distribution function is then $\chi(x - b)$ which has Fourier transform e^{itb}. In particular, $L(t;\sigma) \equiv 1$ belongs to $\sigma = \chi$.

The Fourier transform of the distribution function $\sigma(ax)$ is $L(t/a;\sigma)$. If $\sigma(x) = \sigma(ax)$, where $a > 0$, then $a = 1$ unless $\sigma(x) = \chi(x)$. This necessary condition is sufficient also, i. e., the distribution functions which are similar to $\chi(x)$ are identical with $\chi(x)$.

For a given distribution function σ, put $[\sigma] = \limsup |L(t;\sigma)|$, where $t \to +\infty$ or $t \to -\infty$. It is clear from $|L(t;\sigma)| \leqq 1$ that $0 \leqq [\sigma] \leqq 1$. If σ is absolutely continuous, then $[\sigma] = 0$ in view of the Riemann-Lebesgue lemma. If $[\sigma] = 0$, then σ need not be absolutely continuous. If $[\sigma] = 0$, then $\sigma(x)$ is everywhere continuous. For if σ has at least one discontinuity point, then $L(t;\sigma)$ has a component which is almost periodic in the sense of Bohr, is not identically zero and is not destroyed by the complementary component of $L(t;\sigma)$; in this connection cf. Haviland and Wintner [11]. If σ is purely discontinuous, then $L(t;\sigma)$ is almost periodic in the sense of Bohr, and so $[\sigma] = 1$ in view of $L(0;\sigma) = 1$. It would be interesting to know whether $[\sigma] = 1$ may or may not hold if σ is not purely discontinuous. It may be mentioned that there exists for $\epsilon > 0$ a distribution function σ which is nowhere discontinuous but such that $[\sigma] > 1 - \epsilon$. First, there exists for every positive $a < \frac{1}{2}$ a distribution function σ which is continuous but not absolutely continuous and is defined by the condition

$$L(t;\sigma) = \prod_{k=1}^{\infty} \cos(a^k t);$$

cf. Jessen and Wintner [12], § 6 and Kershner and Wintner [13]. Now choose $a = m^{-1}$, where $m > 2$ is a fixed integer, and put $t = 2\pi m^j$, where $j = \pm 1, \pm 2, \cdots$. Then

$$L(2\pi m^j;\sigma) = \prod_{k=1}^{\infty} \cos(m^{-k} 2\pi m^j) = \prod_{k=1}^{\infty} \cos(2\pi/m^k)$$

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for every j. Hence, on letting $|j| \to \infty$ and keeping $a = m^{-1}$ fixed,

$$[\sigma] \geqq \prod_{k=1}^{\infty} |\cos(2\pi/m^k)|,$$

where σ depends on m. The last product clearly tends to 1 as $m \to \infty$, showing that $[\sigma] > 1 - \epsilon$ for a suitable continuous σ.

One of the reasons for the importance of the Fourier transformation in the theory of distribution functions is the fact that $L(t; \sigma_1 * \sigma_2)$ is more easily obtained than $\sigma_1 * \sigma_2$. In fact,

$$L(t; \sigma_1 * \sigma_2) = L(t; \sigma_1) L(t; \sigma_2); \tag{4}$$

cf. Haviland [9], [10].

A sequence $\{\sigma_n\}$ of distribution functions is convergent if and only if the sequence $\{L(t; \sigma_n)\}$ of the Fourier transforms is uniformly convergent in every fixed finite interval $|t| <$ const., and $\lim L(t; \sigma_n)$ is then $L(t; \lim \sigma_n)$. This is Lévy's "continuity theorem" for the Fourier transforms of distribution functions. The theorem implies an existence statement, namely the assertion that the distribution function $\lim \sigma_n$ exists. A simple proof has recently been given by the present author; cf. Haviland [10], where further references are given.

The infinite convolution $\sigma_1 * \sigma_2 * \cdots$ is said to converge to the distribution function σ if $\sigma_1 * \cdots * \sigma_n \to \sigma$ as $n \to \infty$. Since

$$L(t; \sigma_1 * \cdots * \sigma_n) = L(t; \sigma_1) \cdots L(t; \sigma_n)$$

in view of (4), the continuity theorem implies that $\sigma_1 * \sigma_2 * \cdots$ is a convergent infinite convolution if and only if the infinite product

$$\prod_{k=1}^{\infty} L(t; \sigma_k) \tag{5}$$

is uniformly convergent in every fixed finite interval $|t| <$ const., and that this product is then $L(t; \sigma_1 * \sigma_2 * \cdots)$. For a detailed theory of infinite convolutions cf. Jessen and Wintner [12]. The references given there must be completed by one to the paper [8] of Hausdorff who was apparently the first to consider an infinite convolution in a particular case (cf. § 2 below).

Since two similar distribution functions differ from each other but in the choice of the unit of x, it is clear from the statistical meaning of the convolution process that if σ is a possible universal law of random distribution, then the convolution of any pair of distribution functions which are similar

to σ must be again similar to σ; cf. Bessel [1], Cauchy [4], Pólya [17]. If this condition is satisfied, σ is termed a stable distribution function. Thus the condition of stability is that there exists for every $a > 0$ and every $b > 0$ a $c = c(a, b) > 0$ such that

$$\sigma(x/a) * \sigma(x/b) = \sigma(x/c),$$

i. e.,

$$L(at;\sigma)L(bt;\sigma) = L(ct;\sigma). \tag{6}$$

This does not imply that there exists for every $a > 0$ and every $c > 0$ a $b = b(a, c) > 0$. It is clear from a remark made above with regard to similar distribution functions that if σ is stable, c is uniquely determined by a and b unless $\sigma = \chi$.

2. *Fourier transforms of convex analytic distributions.* The kernel of the representation of the function

$$1/\text{Cosh}\, t \tag{7}$$

as a Fourier cosine transform is (i) positive and (ii) monotone decreasing. This is implied by the fact that (7) is, up to constant factors, self-reciprocal with respect to the Fourier cosine transform. Theorem I will deal with a large class of Fourier transforms satisfying conditions (i) and (ii). The class in question contains each of the functions

$$(it)^\nu/J_\nu(it), \text{ where } \nu > -1. \tag{8}$$

The example (8) implies (7), since $\nu = -\frac{1}{2}$ is not excluded; for $\nu = \frac{1}{2}$ one obtains $t/\text{Sinh}\, t$. The treatment of the general function class in question will be based on an infinite convolution, considered by Hausdorff [8] in the explicitly available case (7). For the treatment of the general case, which is not self-reciprocal, an existence theorem is needed. Although a theorem to this effect might easily be proven directly, this will not be done, since the existence statement in question is implied by the continuity theorem mentioned in § 1.

Let $f(z)$ be an even transcendental entire function which has but real zeros and is positive at $z = 0$. Suppose further that the entire function $f(\sqrt{z})$ is of genus 0. Thus

$$A = \sum_{k=1}^{\infty} a_k^2 < +\infty, \qquad (a_1 \geqq a_2 \geqq \cdots; a_k > 0), \tag{9}$$

where

$$\pm 1/a_1, \pm 1/a_2, \cdots \tag{10}$$

denote the zeros of $f(z) = f(-z)$, and

$$(11) \qquad f(z) = f(0) \prod_{k=1}^{\infty} (1 - a_k^2 z^2),$$

where $f(0) > 0$. In particular,

$$(12) \qquad \max_{|z| \leqq |t|} |f(z)| = f(it),$$

where $-\infty < t < +\infty$. Examples of functions $f(z)$ which satisfy the conditions just mentioned are infinite products like

$$f(z) = \prod_{n=1}^{\infty} \cos(b_n z) \text{ and } f(z) = \prod_{n=1}^{\infty} J_0(b_n z), \quad (\sum_{n=1}^{\infty} b_n^2 < +\infty; \; b_n > 0),$$

(occurring in connection with the simplest types of infinite convolutions), the trigonometrical integrals

$$f(z) = \int_0^1 \psi(x) \cos zx dx$$

investigated by Pólya [16] and, in particular, the functions

$$f(z) = z^{-\nu} J_\nu(z), \text{ where } \nu \geqq -\tfrac{1}{2}.$$

All these examples $f(z)$ admit of a representation of the form

$$(13) \qquad f(z) = \int_0^{+\infty} \cos zx d\phi(x),$$

where $\phi(x)$ is monotone, bounded and not everywhere constant. On the other hand, not every $f(z)$ under consideration is representable in the form (13). In fact, (13) implies (12), hence it also implies

$$(14) \qquad \max_{|z| \leqq |t|} |f(z)| > C e^{c|t|},$$

where C and c are positive constants. Now suppose that the zeros (10) of $f(z)$ are so scarce as to make the series $a_1 + a_2 + \cdots$ convergent. Then

$$h(z) = \prod_{k=1}^{\infty} (1 - a_k z)$$

is a canonical product of order 0. Since $f(z) = f(0)h(z)h(-z)$ in view of (11), the function $f(z)$ also is of order 0, hence cannot satisfy the condition (14) which is a necessary condition for (13). In the following existence theorem it is not assumed that $f(z)$ is representable in the form (13).

THEOREM I. *Let $f(z)$, where $f(0) > 0$, be an even transcendental entire*

function which is of genus 0 in z^2 and has but real zeros. Then there exists a distribution function $\sigma(x)$ which is convex in the sense defined in § 1, has for $-\infty < x < +\infty$ bounded derivatives of arbitrarily high order and is connected with f by the formula

(15) $$f(0)/f(it) = L(t;\sigma), \text{ where } -\infty < t < +\infty,$$

L being the Fourier transform (1); and so $L(t;\sigma) = 2\int_0^{+\infty} \cos tx d\sigma(x)$ in view of the symmetry of σ.

If $f(z)$ satisfies the additional condition (14), then the distribution function $\sigma(x)$ implicitly defined by (15) is regular and bounded in a strip $-\alpha < y < \alpha$ of the z-plane, where $z = x + iy$.

Remark. The example $f(z) = \cos z$ shows that σ need not be an entire function. In fact, (7) is, up to constant factors, self-reciprocal and has therefore poles on the boundary of a strip of finite width. Also, the function $f(0)/f(iz)$ represented for real z by (15) always has on the imaginary axis a pole in a finite distance from $z = 0$.

Proof. Let $\gamma = \gamma(x)$ denote distribution function

$$\gamma(x) = \tfrac{1}{2}\int_{-\infty}^{x} e^{-|y|} dy.$$

Since the derivative $\gamma'(x) = \gamma'(-x)$ is a decreasing function of $|x|$, the distribution function γ is convex in the sense of § 1. Furthermore,

$$L(t;\gamma) = \int_{-\infty}^{+\infty} e^{itx}\, \tfrac{1}{2} e^{-|x|}\, dx = \int_0^{+\infty} \cos tx\, e^{-x}\, dx,$$

i. e., $L(t;\gamma) = (1 + t^2)^{-1}$. Hence on denoting by $\sigma_k(x)$ the distribution function $\gamma(x/a_k)$,

$$L(t;\sigma_k) = L(a_k t;\gamma) = (1 + a_k^2 t^2)^{-1},$$

and so the product (5) is, in view of (9), uniformly convergent in every fixed finite interval $|t| <$ const. Consequently, there exists a distribution function σ represented by the infinite convolution $\sigma_1 * \sigma_2 * \cdots$, and

$$L(t;\sigma) = \prod_{k=1}^{\infty} L(t;\sigma_k) = \prod_{k=1}^{\infty} (1 + a_k^2 t^2)^{-1},$$

which proves (15) in view of (11). Since $a_k > 0$,

$$L(t;\sigma) = [\prod_{k=1}^{\infty} (1 + a_k^2 t^2)]^{-1} = O\ (|t|^{-N}),\ t \to \pm\infty,$$

holds for every fixed N. Hence $\sigma(x)$ has for $-\infty < x < +\infty$ bounded derivatives of arbitrarily high order. This follows from the last estimate, where N may be arbitrarily large, by successive differentiation of (2). Since $\gamma(x)$ is convex, so is $\gamma(x/a_k) = \sigma_k(x)$ for every k; hence $\sigma_1 * \cdots * \sigma_m$ and therefore

$$\lim_{m=\infty} \sigma_1 * \cdots * \sigma_m = \sigma_1 * \sigma_2 * \cdots = \sigma$$

also is convex. Finally, if $f(z)$ satisfies the additional condition (14), then it is clear from (12) and (15) that (3) is satisfied for $\lambda = 1$ and $a = c > 0$. This completes the proof of Theorem I.

Since σ is convex, hence symmetric,

$$\int_{-\infty}^{+\infty} f(x)\,d\sigma(x) = 2\int_{0}^{+\infty} f(x)\,d\sigma(x) \quad \text{or} \quad = 0$$

according as $f(x)$ is even or odd, provided that the latter integral is convergent. Information about the behavior of σ at $x = +\infty$, hence at $x = -\infty$, is given by

THEOREM II. *Let $f(z)$ satisfy the general requirements of Theorem I but not necessarily the additional condition* (14), *and let σ be the distribution function defined by* (15). *Then all integrals*

$$(16) \qquad \int_{0}^{+\infty} x^n\,d\sigma(x) \qquad (n = 0, 1, 2, \cdots)$$

are convergent and belong to a determined Stieltjes moment problem. Furthermore,

$$(17) \qquad \int_{x}^{+\infty} d\sigma(y) = O\,(\exp\{-cx^{\frac{2}{3}}\}) \text{ as } x \to +\infty,$$

where $c > 0$.

Remark. The estimate leading to (17) is so crude that it does not attach any particular significance to the numerical value $\frac{2}{3}$ of the exponent of x. On the other hand, (17) describes the true situation up to the numerical value of this exponent. For if $f(z) = \cos z$, it is clear from the example (7) that (17) becomes false if one replaces $\frac{2}{3}$ by $1 + \epsilon$. Thus the best value of the exponent is somewhere between $\frac{2}{3}$ and 1, and it is not proven that it is less than 1. That it cannot be greater than 1, agrees with the fact that $1/f(z)$ is not an entire function.

Proof. Put

$$\mu_r(\psi) = \int_{-\infty}^{+\infty} x^r\,d\psi(x), \qquad (r = 1, 2, \cdots),$$

and let γ and σ_k denote the distribution functions defined in the Proof of Theorem I. Thus $\mu_{2m+1}(\gamma) = 0$, while

$$\mu_{2m}(\gamma) = \int_{-\infty}^{+\infty} x^{2m}\, \tfrac{1}{2} e^{-|x|} dx = (2m)\,!;$$

hence, since $\sigma_k(x) = \gamma(x/a_k)$,

$$\mu_{2m}(\sigma_k) = a_k^{2m}\mu_{2m}(\gamma) = (2m)\,!\, a_k^{2m}.$$

Since, by Stirling's formula, $N! < (\alpha N)^N$ for every positive integer N and for some constant $\alpha > 0$, it follows that

$$\mu_{2m}(\sigma_k) < (2\alpha m\, a_k)^{2m}.$$

Now

$$\mu_{2m}(\sigma_1 * \cdots * \sigma_n) \leqq m^m \Sigma\, C_{hj\ldots}\; \mu_{2h}(\sigma_1)\mu_{2j}(\sigma_2) \cdots,$$

where $C_{hj\ldots}$ denotes the multinomial coefficient

$$C_{hj\ldots} = (h + j + \cdots)\,!\,/(h\,!\,j\,! \cdots)$$

and the summation Σ runs through all collections of n non-negative integers $h, j, \cdots$ for which $h + j + \cdots = m$; for proof cf. Wintner [28], where reference is given to a similar inequality of Paley and Zygmund. Thus

$$\mu_{2m}(\sigma_1 * \cdots * \sigma_n) < m^m \Sigma\, C_{hj\ldots}\; (2\alpha\, h\, a_1)^{2h}(2\alpha\, j\, a_2)^{2j} \cdots,$$

or, since $h + j + \cdots = m$,

$$\mu_{2m}(\sigma_1 * \cdots * \sigma_n) < m^m(4\alpha^2)^m \Sigma\, C_{hj\ldots}\, a_1^{2h}a_2^{2j} \cdots (h^h j^j \cdots)^2.$$

On combining this inequality with the crude estimate

$$h^h j^j \cdots \leqq m^h m^j \cdots = m^m$$

and with the multinomial theorem, i. e., with the identity

$$\Sigma\, C_{hj\ldots}\, a_1^{2h}a_2^{2j} \cdots = (a_1^2 + \cdots + a_n^2)^m,$$

it is seen that

$$\mu_{2m}(\sigma_1 * \cdots * \sigma_n) < m^m(4\alpha^2)^m(a_1^2 + \cdots + a_n^2)^m(m^m)^2.$$

Consequently, from (9),

$$\mu_{2m}(\sigma_1 * \cdots * \sigma_n) < C^m m^{3m},$$

where $C = 4\alpha^2 A$ is independent both of n and m. On placing $\rho_n = \sigma_1 * \cdots * \sigma_n$ and letting $n \to \infty$ for a fixed m, Helly's theorem on term-by-term integration shows that

$$\int_{-R}^{R} x^{2m} d\rho_n(x) \to \int_{-R}^{R} x^{2m} d\sigma(x)$$

for every fixed $R > 0$; for $\rho_n = \sigma_1 * \cdots * \sigma_n$ tends, as $n \to \infty$, to the infinite convolution $\sigma_1 * \sigma_2 * \cdots$ which defined σ in the Proof of Theorem I. On letting $R \to +\infty$ for a fixed m, it follows that

$$(18) \qquad \mu_{2m}(\sigma) \leqq C^m m^{3m}$$

for every m. Now if M_n denotes the integral (16) and if every M_{2m} is convergent, then so is every M_{2m+1} in view of the Schwarz inequality; and every M_{2m} is convergent in virtue of (18), since $\mu_{2m} = 2M_{2m}$. It also follows from (18) that $(M_{2m})^{-1/(4m)} \geqq \text{const.}/m^{3/4}$, which implies the divergence of the series

$$\sum_{m=1}^{\infty} (M_m)^{-1/(2m)}.$$

Hence the criterion of Carleman [3], p. 81, for the Stieltjes case shows that (16) belongs to a determined Stieltjes moment problem.

The transition from (18) to (17) requires but a standard argument (cf. Zygmund [30], p. 124). In fact, since

$$\int_0^{+\infty} x^{2m} d\sigma(x) = \tfrac{1}{2}\mu_{2m}(\sigma) \text{ and } \int_0^{+\infty} d\sigma(x) = \tfrac{1}{2},$$

Hölder's inequality shows that

$$\int_0^{+\infty} x^{2m/3} d\sigma(x) \leqq \left[\int_0^{+\infty} (x^{2m/3})^3 d\sigma(x)\right]^{1/3}\left[\int_0^{+\infty} 1^{3/2} d\sigma(x)\right]^{2/3} < [\mu_{2m}(\sigma)]^{1/3}.$$

Thus it is seen from (18) that

$$\int_0^{+\infty} x^{2m/3} d\sigma(x) < (Bm)^m,$$

where $B = C^{1/3}$ and $m = 1, 2, \cdots$. Hence it is clear from Stirling's formula,

$$m!^{-1}(mB)^m \sim (Be)^m/(2\pi m)^{1/2}, \quad m \to \infty,$$

that the power series

$$p(z) = \sum_{m=0}^{\infty} m!^{-1} \int_0^{+\infty} x^{2m/3} d\sigma(x) \cdot z^m$$

has a non-vanishing radius of convergence. Since the coefficients of this power series are positive, it follows by writing y instead of x that for every sufficiently small $z = c > 0$

$$\int_0^{+\infty} \exp(cy^{2/3})\,d\sigma(y) = p(c) < +\infty.$$

Consequently, if $x > 0$,

$$\exp(cx^{2/3}) \int_x^{+\infty} d\sigma(y) \leqq \int_x^{+\infty} \exp(cy^{2/3})\,d\sigma(y) \leqq p(c) = \text{const.}$$

This completes the proof of Theorem II.

Let Ξ be defined by $\Xi(z) = \xi(\tfrac{1}{2} + iz)$, where $\xi(z)$ denotes the Riemann ξ-function; cf. Pólya [19], Titchmarsh [20], p. 43.

THEOREM III. *On Riemann's hypothesis,*

$$\Xi(0)/\Xi(it) = L(t;\sigma), \quad -\infty < t < +\infty,$$

where $\sigma(x)$ is a convex distribution function which is regular and bounded in a strip $-\alpha < y < \alpha$.

Proof. According to the Hadamard theory, the entire function $\Xi(\sqrt{z})$ is of genus 0 in z. Furthermore, the Riemann integral representation of $\Xi(z)$ is of the form (13), where $\phi'(x) > 0$. Finally, Riemann's hypothesis is that all zeros of $\Xi(z)$ are real. Hence Theorem III is implied by Theorem I.

THEOREM IV. *Independently of the Riemann hypothesis,*

$$\Xi(t)/\Xi(0) = L(t;\sigma), \quad -\infty < t < +\infty,$$

where $\sigma(x)$ is a convex distribution function which is regular and bounded in a strip $-\alpha < y < \alpha$.

For the proof of the convexity of σ cf. Wintner [27]. The Riemann integral representation of Ξ shows that the least upper bound of α is $\pi/8$ and that the lines $y = \pm\pi/8$ form a natural boundary of σ. The point in Theorem IV is that σ is convex. This implies the following result which is less deep and is of an older date, since it has been pointed out in 1916 in an equivalent form by Wilton [22].

COROLLARY. *On denoting by $\mathrm{Z}(s)$ the meromorphic function $\zeta(s)\Gamma(\tfrac{1}{2}s)\pi^{-s/2}$,*

$$\mathrm{Z}(\tfrac{1}{2} + it)/\mathrm{Z}(\tfrac{1}{2}) = L(t;\rho), \quad -\infty < t < +\infty,$$

where ρ is a convex distribution function.

This is a consequence of Theorem IV, but not conversely. First,

$$\Xi(t) = -\tfrac{1}{2}(\tfrac{1}{4} + t^2)Z(\tfrac{1}{2} + it)$$

by the Riemann definition of $\Xi(t) = \xi(\frac{1}{2} + it)$; cf. Titchmarsh [20], p. 43. Put $\delta(x) = \gamma(\frac{1}{2}x)$, where $\gamma(x)$ is the convex distribution function occurring in the proof of Theorem I, so that $L(t;\gamma) = (1 + t^2)^{-1}$. Thus δ is a convex distribution function. Hence, if σ denotes the distribution function occurring in Theorem IV, then $\rho = \delta * \sigma$ also is convex. Now, on using (4),

$$L(t;\rho) = L(t;\delta)L(t;\sigma) = L(2t;\gamma)\Xi(t)/\Xi(0),$$

where $L(2t;\gamma) = (1 + (2t)^2)^{-1}$; hence

$$L(t;\rho) = (1 + 4t^2)^{-1}[-\tfrac{1}{8}(1 + 4t^2)Z(\tfrac{1}{2} + it)]/\Xi(0) = -\tfrac{1}{8}Z(\tfrac{1}{2} + it)/\Xi(0).$$

Consequently,

$$Z(\tfrac{1}{2} + it) = Z(\tfrac{1}{2})L(t;\rho),$$

where ρ is convex; q. e. d. It may be mentioned that the Fourier transform $L(t;\rho)$ is precisely the one which occurs in Hardy's proof for the existence of infinitely many real zeros of $\zeta(\frac{1}{2} + it)$.

The absolutely convergent product (19) considered in the next Theorem is often mentioned in the elementary theory of canonical products; cf., e. g., Francis and Littlewood [5], p. 5. Although the zeros of the entire function (19), which is of genus 0 in t^2, are all real and equidistant, the rapid increase of the multiplicity of these zeros clearly puts the function beyond Pringsheim's "normal type of order 1." Apparently it is not known whether or not the function is related to solutions of standard linear differential equations of the second order.

THEOREM V. *There exists a convex distribution function* τ *such that*

$$\prod_{k=1}^{\infty} \cos(t/k) = L(t;\tau), \tag{19}$$

and $\tau(x)$ *is regular and bounded in a strip* $-\alpha < y < \alpha$. *Furthermore, there exists a* $c > 0$ *such that*

$$\tau(x) = 1 - \tau(-x) = O(\exp\{-cx^2\}) \text{ as } x \to -\infty$$

and Carleman's series

$$\sum_{m=1}^{\infty} (\mu_{2m})^{-1/(2m)}, \quad \text{where } \mu_{2m} = \int_{-\infty}^{+\infty} x^{2m} d\tau(x) < +\infty,$$

is divergent.

For proof cf. Wintner [28], and [29], p. 838. The least upper bound of the admissible values of the width 2α is not less than 1; cf. Jessen and Wintner [12], p. 62. It is not known if the boundary of the widest strip is a natural boundary of τ. Nor is it known that the widest strip has a boundary; it is not even proven that τ is not an entire function. A simple proof for the existence of an $\alpha > 0$ proceeds as follows. Let

$$a_1 > 0, a_2 > 0, \cdots \text{ and } a_1^2 + a_2^2 + \cdots < +\infty$$

and let k be so large that $a_k \mid t \mid < 1$, where $\mid t \mid \neq 0$ is fixed. Since there exist a positive constant $C < 1$ and a positive constant B such that

$$0 < \cos \vartheta < 1 - C\vartheta^2 \text{ and } \log (1 - \vartheta) < -B\vartheta, \text{ where } 0 < \vartheta < 1,$$

it is clear that

$$0 < \prod_{a_k|t|<1} \cos (a_k t) < \prod_{a_k|t|<1} (1 - Ca_k^2 t^2) < \exp \sum_{a_k|t|<1} -BCa_k^2 t^2;$$

hence

$$(20) \qquad |\prod_{k=1}^{\infty} \cos (a_k t)| \leqq \prod_{a_k|t|<1} \cos (a_k t) = \exp O(-At^2 \sum_{a_k|t|<1} a_k^2),$$

where $A = BC > 0$ and $t \to \pm \infty$. On choosing $a_k = k^{-1}$, it follows from (20) that (19) satisfies (3) with $\lambda = 1$. As far as the convexity of τ is concerned, cf. (35) below.

Remark. It is instructive to contrast the distribution functions σ and τ occurring in Theorems IV and V. The relation (19) means that $\tau(x)$ is the convolution of the infinite sequence $\beta(x), \beta(2x), \beta(3x), \cdots$ of Bernoulli distribution functions, where $\beta(x) = 0, \frac{1}{2}$ or 1 according as x lies on the left, in the interior or on the right of the interval $-1 < x < 1$. This *statistical factorization* of τ yields for the entire function $L(t;\tau)$ the factorization (19) which, being *not* the canonical factorization of Weierstrass-Hadamard, puts the reality of all zeros of $L(t;\tau)$ into evidence. In Theorem IV, the reality of all zeros of $L(t;\sigma)$ is Riemann's hypothesis. Thus one should like to obtain instead of the Weierstrass-Hadamard factorization of $L(t;\sigma)$ a factorization (5) of $L(t;\sigma)$ into a product of Fourier transforms $L(t;\sigma_k)$ or, what is the same thing, a statistical decomposition of σ into an infinite convolution $\sigma_1 * \sigma_2 * \cdots$, based on particular statistical and not on general function-theoretical properties. A quantitative illustration of the possible analogy between $L(t;\sigma)$ and $L(t;\tau)$ may be obtained as follows. Let $N_0(T)$ be the number of zeros of $L(t;\sigma) = \Xi(t)/\Xi(0)$ in the interval $0 < t < T$ and let $N'(t)$ denote the corresponding function belonging to $L(t;\tau)$, each

zero being counted according to its multiplicity. The Riemann-Mangoldt asymptotic formula implies that $N_0(T) \sim (2\pi)^{-1}T \log T$ on Riemann's hypothesis. This relation, although so far unproven, means very much less than Riemann's hypothesis and is possibly the estimate which Riemann had in mind in his famous statement: "Man findet . . . *etwa* . . . so viel reelle Wurzeln . . ." (the italics are mine). On the other hand, $N'(T)$ is the number of zeros of

$$\prod_{k=1}^{[2T/\pi]} \cos(t/k)$$

between $t = 0$ and $t = T$; for if $k > 2T/\pi$, then $T/k < \frac{1}{2}\pi$, and so the factor $\cos(t/k)$ in (19) does not contribute to $N'(T)$. Since $\cos t$ has between $t = 0$ and $t = T$ about $\pi^{-1}T$ zeros, it follows that

$$N'(T) \sim \sum_{k=1}^{[2T/\pi]} \pi^{-1}T/k \sim \pi^{-1}T \log T.$$

Hence $N_0(T) \sim \frac{1}{2}N'(T)$ on Riemann's hypothesis. The distance between two subsequent discontinuity points of $N_0(T)$ tends, according to Littlewood, on Riemann's hypothesis to zero (cf. Titchmarsh [20], p. 60), while the distance between two subsequent discontinuity points of $N'(T)$ has the constant value π. Thus $N_0(T) \sim \frac{1}{2}N'(T)$ agrees with the rapid increase of the multiplicities of the zeros of $L(t;\tau)$.

If p_k denotes the k-th prime number,

$$\prod_{k=1}^{\infty} \cos(t/p_k) \tag{21}$$

also is the Fourier transform L of a distribution function which has derivatives of arbitrarily high order along the real axis; cf. Wintner [26]. The latter statement is clear from (20) also. In fact, $a_k = p_k^{-1}$, so that (20) implies that (3) is satisfied for every $\lambda < 1$. It is not known whether or not the distribution function belonging to (21) is analytic,[1] and the question of convexity also is unanswered.

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[1] On the other hand, it is easy to prove that the infinite convolution in question is *a distribution function $\sigma(x)$ which is, for $-\infty < x < +\infty$, quasi-analytic in the sense of Denjoy and Carleman.* First,

$$L(t;\sigma) = O(\exp - At^2 \sum_{p_n > |t|} p_n^{-2}), \quad t \to \pm\infty, \quad A > 0,$$

in view of (20). Hence it is clear from the prime-number theorem ($p_n \sim n \log n$) or even from the elementary inequalities of Tchebycheff that

$$L(t;\sigma) = O(\exp - C\,|t|/\log|t|), \quad t \to \pm\infty, \tag{I}$$

The simplest example of Theorem I was the standard formula expressing the fact that (7) is, up to constant factors, a self-reciprocal function of the Fourier cosine transform. Correspondingly, the formula

$$(22) \qquad |\Gamma(b+\tfrac{1}{2}it)|^2 = C_b \int_{-\infty}^{+\infty} (\operatorname{Cosh} x)^{-2b} e^{itx} dx; \ b > 0, \ C_b = 2^{1-2b}\Gamma(2b),$$

which generalizes the previous one from $b = \frac{1}{2}$ to an arbitrary $b > 0$ and also is standard (cf., e. g., Mathias [15], p. 113), is but the simplest illustration of a general theorem:

THEOREM VI. *If the entire function $g(z)$ is real along the real axis, has infinitely many zeros which are all zero or negative and is of genus 0 or 1, then there exists for every $b > 0$ a convex distribution function $\sigma = \sigma_b = \sigma_b(x)$ such that*

$$(23) \qquad |g(b+it)|^{-2} = |g(b)|^{-2} L(t;\sigma_b); \ -\infty < t < +\infty.$$

Furthermore, this distribution function has derivatives of arbitrarily high order for $-\infty < x < +\infty$ and its behavior at $x = \infty$ is the same as in Theorem II. *If in addition the zeros of $g(z)$ are so abundant and so regularly situated as to imply a certain inequality analogous to the additional condition* (14) *of Theorem* I, *then σ_b is regular and bounded in a strip $-\alpha < y < \alpha$.*

Remark. If

$$g(z) = 1/\Gamma(z) = ze^{Cz}\prod_{m=1}^{\infty}(1+z/m)e^{-z/m},$$

the genus of $g(z)$ is 1 and the zeros $z = -m < 0$ are sufficiently abundant and "regularly distributed." Thus (22), where $g(z) = 1/\Gamma(\frac{1}{2}z)$, is an explicit example of Theorem VI. In fact, $(\operatorname{Cosh} x)^{-2b}$ is for every $b > 0$ a positive decreasing function of $|x|$. Since $(\operatorname{Cosh} z)^{-2b}$ has on the boundary of a strip $-\alpha < y < \alpha$ of width $2\alpha = \pi$ poles $(2b = 1, 2, \cdots)$ or logarithmical singularities $(2b \neq 1, 2, \cdots)$, the example (22) also shows that the last statement

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where C is a positive constant. Now it is easy to see that

$$(\text{II}) \qquad \int_e^{+\infty} t^{n-1} \exp(-Ct/\log t)\,dt < (\text{const.}\, n \log n)^n.$$

This follows from Stirling's formula. According to a remark of Mr. R. B. Kershner, one obtains a short proof of (II) by introducing instead of t the integration variable $t/\log t$. Now it is obvious from (I) and (II) by successive differentiation of (2) that

$$|\sigma^{(n)}(x)| < (Mn \log n)^n,$$

where M is independent both of n and x. Since $\Sigma(n \log n)^{-1}$ is a divergent series, the quasi-analyticity of $\sigma(x)$ follows by the Denjoy-Carleman criterion; cf. Carleman [3].

of Theorem VI cannot be improved to the statement that σ_b is an entire function.

Proof. If Theorem VI is true on the assumption $g(0) \neq 0$, it is true also when $g(0) = 0$. For if $g(z)$ vanishes at $z = 0$ in the order $d \geqq 1$, there exists, by assumption, a convex distribution function σ_b such that

$$| z^{-d} g(z) | = \text{const. } L(t; \sigma_b), \text{ where } z = b + it.$$

Thus, since $L(0; \sigma_b) = 1$,

$$| g(b + it) |^{-2} = | g(b) |^{-2} L(t; \sigma_b) (1 + t^2/b^2)^{-d},$$

where $g(b) \neq 0$, since $b > 0$. Now if $\gamma(x)$ denotes the convex distribution function occurring in the Proof of Theorem I and if $\gamma_b(x)$ is the similar distribution function $\gamma(bx)$, then $L(t; \gamma) = (1 + t^2)^{-1}$, hence

$$(1 + t^2/b^2)^{-d} = (L(t/b; \gamma))^d = (L(t; \gamma_b))^d = L(t; \gamma_{bd}),$$

where γ_{bd} denotes the distribution function $\gamma_b * \cdots * \gamma_b$ (d times). Thus γ_{bd} is a convex distribution function and

$$| g(b + it) |^{-2} = | g(b) |^{-2} L(t; \sigma_b) L(t; \gamma_{bd}).$$

Now $L(t; \sigma_b) L(t; \gamma_{bd}) = L(t; \sigma_b * \gamma_{bd})$, and $\sigma_b * \gamma_{bd}$ also is a convex distribution function. Consequently, the case $g(0) = 0$ is reducible to the case $g(0) \neq 0$.

Now let $g(0) \neq 0$. Consider first the case where the genus of the canonical product belonging to $g(z)$ is 1. Then, since all zeros are negative and $g(x)$ is real,

$$g(z) = e^{Az} \prod_{m=1}^{\infty} (1 + r_m z) e^{-r_m z}$$

where

$$A \gtreqless 0, \; r_m > 0 \text{ and } \sum_{m=1}^{\infty} r_m{}^2 < +\infty.$$

Hence, on placing $z = b + it$, where $b > 0$ is fixed and $-\infty < t < +\infty$,

$$| g(b + it) |^2 = e^{2Ab} \prod_{m=1}^{\infty} [(1 + r_m b)^2 + r_m{}^2 t^2] e^{-2r_m b}.$$

This may be written in the form

$$(24) \qquad | g(b + it) |^{-2} = | g(b) |^{-2} \prod_{m=1}^{\infty} (1 + s_m{}^2 t^2)^{-1},$$

where

$$s_m = r_m / (1 + r_m b) > 0,$$

hence

$$\sum_{m=1}^{\infty} s_m{}^2 < +\infty, \text{ since } \sum_{m=1}^{\infty} r_m{}^2 < +\infty.$$

Thus it is clear that the arguments used in the Proof of Theorems I and II are applicable without change to the product (24). This proves Theorem VI for the case where the genus of the canonical product occurring in $g(z)$ is 1. If this genus is 0, then

$$g(z) = e^{Az} \prod_{m=1}^{\infty} (1 + r_m z),$$

where

$$A \gtreqqless 0, \; r_m > 0 \text{ and } \sum_{m=1}^{\infty} r_m < +\infty.$$

Thus

$$| g(b + it) |^2 = e^{2Ab} \prod_{m=1}^{\infty} [(1 + r_m b)^2 + r_m{}^2 t^2].$$

Hence, on defining s_m by the same formula as in the previous case, it is easy to see that (24) is again valid. This completes the proof of Theorem VI.

The next Theorem will be an extension of Theorem I to the case of an "erhöht" genus, the latter being meant in the sense of Pólya [19].

THEOREM VII. *Let $F(z)$ be an entire function of the form*

$$F(z) = \exp(-bz^2) f(z),$$

where b is a real non-negative constant and $f(z)$ satisfies the general requirements of Theorem I *but not necessarily the additional condition* (14). *Then if $b > 0$, there exists a convex distribution function $\rho(x)$ such that*

$$(25) \qquad F(0)/F(it) = L(t;\rho), \; -\infty < t < +\infty,$$

and ρ is a transcendental entire function which is bounded in every strip $-\alpha \leqq y \leqq \alpha$.

Proof. Theorem I belongs to $b = 0$. If $b > 0$, on replacing t by $b^{-1/2}t$ it may be assumed that $b = 1$. Thus

$$(26) \qquad F(0)/F(it) = \exp(-t^2) f(0)/f(it) = \exp(-t^2) L(t;\sigma),$$

where σ is by Theorem I a convex distribution function. Since the even function $\exp(-x^2)$ is, up to constant factors, self-reciprocal under the Fourier cosine transform and decreases as $|x|$ increases, there exists a convex distribution function $\omega(x)$ such that $\exp(-t^2) = L(t;\omega)$, this distribution

being a symmetric Gaussian or normal distribution function. On placing $\rho = \omega * \sigma$, the distribution function ρ also is convex. Furthermore, ρ satisfies (25) in view of (26) and (4). Finally, since $| L(t;\sigma) | \leqq 1$ for every σ and for every t,

$$| L(t;\rho) | = | L(t;\omega) | \, | L(t;\sigma) | \leqq \exp(-t^2),$$

and so $L(t;\rho)$ satisfies (3) for a $\lambda > 1$. This completes the proof of Theorem VII.

The proof depended on the fact that if $\lambda = 2$, there exists a symmetric distribution function $\omega_\lambda = \omega_\lambda(x)$ such that

$$(27) \qquad \exp(-| t |^\lambda) = L(t;\omega_\lambda), \quad -\infty < t < +\infty,$$

and that this $\omega = \omega_2$, being a symmetric Gaussian distribution function, is convex. According to Lévy [14], there exists a distribution function ω_λ satisfying (27) also when $0 < \lambda < 2$. If $\lambda = 1$, then one obtains the arcustangent distribution function, while if $0 < \lambda < 1$ or $1 < \lambda < 2$, then ω_λ is not an elementary function. All these distribution functions are symmetric, since $L(t;\omega_\lambda)$ is a real function. Now it will be shown that the distribution function ω_λ is convex not only in the trivial cases $\lambda = 1$ and $\lambda = 2$ but in the cases $0 < \lambda < 1$ and $1 < \lambda < 2$ also. The proof will require a modification of Lévy's generating distribution functions and will yield both the existence and the convexity of ω_λ.

THEOREM VIII. *The condition* (27) *defines for every positive* $\lambda \leqq 2$ *a convex distribution function* ω_λ.

Remark. Since there exists for every $\lambda > 0$ a function $\omega_\lambda(x)$ the derivative of which is

$$(28) \qquad \omega'_\lambda(x) = \pi^{-1} \int_0^{+\infty} \exp(-t^\lambda) \cos xt dt,$$

it is clear from (1) and (2) that (27) may be satisfied by a real even function ω_λ also when $\lambda > 2$. If, however, $\lambda > 2$, then (28) attains (cf. Pólya [17], [18]) negative values for some real x; hence $\omega_\lambda(x)$ is not monotone nondecreasing and therefore not a distribution function. This fact will follow also from a more general result, to be proven later (Theorem XIV). Thus the point in Theorem VIII is that the distribution function ω_λ is convex whenever it exists. This fact will be the basis of § 3.

Information with regard to the analytic character of the distribution

functions ω_λ is given by the next Theorem. Theorems VIII and IX will be proven in § 5, where both appear as particular cases of more general results.

THEOREM IX. *The even function $\omega_\lambda'(x)$ defined by (28) for $\lambda > 0$ has for $-\infty < x < +\infty$ derivatives of arbitrarily high order and is regular at every real $x \neq 0$. If $0 < \lambda < 1$, there is a singularity at $x = 0$, which must be a transcendental singularity, since all derivatives exist at $x = 0$ also, if x is restricted to the real axis. Furthermore, $x = 0$ is a logarithmical branch point, if λ is sufficiently small. Thus if $0 < \lambda < 1$, the behavior of the function at $x = 0$ is about the same as that of Cauchy's standard example,* $\exp\{(\pm x)^\lambda\}$. *Correspondingly, it is not stated that the function elements $\omega_\lambda'(x)$ and $\omega_\lambda'(-x)$, which are regular for $0 < x < +\infty$, are analytic continuations of each other, although $\omega_\lambda'(x)$ is an even function. If, however, $\lambda = 1$, then ω_λ' is a rational function with two purely imaginary simple poles, while if $\lambda > 1$, then ω_λ' is a transcendental entire function of order $(1-\lambda^{-1})^{-1}$.*

Remark. It is not known whether or not (3) implies, if $\lambda < 1$, the analyticity of $\sigma(x)$ at every inner point of the spectrum of $\sigma(x)$ at least in the simplest cases of infinite conventions; cf. Wintner [24], [25], Kershner and Wintner [13], and, in a multidimensional case, Jessen and Wintner [12], Theorems 14 and 19. Thus the analyticity of ω_λ for $x > 0$ is not obvious if $0 < \lambda < 1$.

The relation (27), when interpreted as an estimate of $L(t;\omega_\lambda)$ for large t, does not indicate what is the spectrum of ω_λ. For if (3) is satisfied, then the spectrum of σ need not be the whole line $-\infty < x < +\infty$ but may also be a bounded set. This holds for every fixed $\lambda < 1$, as shown by examples; cf. Wintner [25], [29]. Theorem IX implies, however,

COROLLARY I. *The spectrum of every distribution function ω_λ is the whole line $-\infty < x < +\infty$.*

For suppose the contrary. Then, since the spectrum always is a closed set, there exists an interval (a, b) such that $\omega_\lambda(x) = \text{const.}$, if $a < x < b$. Since $\omega_\lambda'(x)$ is regular both for $-\infty < x < 0$ and $0 < x < +\infty$ and since $\omega_\lambda'(x) = \omega_\lambda'(-x)$, it follows that $\omega_\lambda(x)$ is constant both for $-\infty < x < 0$ and $0 < x < +\infty$. Hence the spectrum either is empty or consists of the single point $x = 0$. The first case is impossible, since the total variation of every distribution function is 1. The second case is possible only for the unit distribution function χ, hence not for ω_λ. This contradiction proves Corollary I.

If $0 < \lambda \leqq 2$, then $\omega_\lambda(x)$ is a distribution function, hence $\omega_\lambda'(x) \geqq 0$,

and so (28) cannot have a real zero of odd multiplicity. It has been pointed out by Pólya ([18], p. 187, statement (a), where "finite" means "at most finite") that if $1 \leqq \lambda \leqq 2$, then (28) cannot have infinitely many real zeros. This result may be improved in two directions, as shown by

COROLLARY II. *If $0 < \lambda \leqq 2$, the function* (28) *has no real zeros.*

This fact, which is completed by the Remark which follows Theorem VIII, may be proven as follows. Suppose that there does exist a real $x = x_0$ such that $\omega_\lambda'(x_0) = 0$. Since $\omega_\lambda(x)$ is a distribution function, $\omega_\lambda'(x) \geqq 0$ for every x. Furthermore, if $|x'| < |x''|$, then $\omega_\lambda'(x') \geqq \omega_\lambda'(x'')$ in virtue of Theorem VIII. Consequently, $\omega_\lambda'(x) = 0$ for every x which is not in the finite interval $-|x_0| < x < |x_0|$. This consequence of the assumption $\omega_\lambda'(x_0) = 0$ contradicts Corollary I and proves therefore Corollary II.

Theorems VIII and IX will be proven in § 5, where the proofs are given for the case of more general integrals which depend on Bessel functions. The following remarks also concern Bessel functions but lie in another direction and are more elementary in character.

THEOREM IX. *There exist for every $\nu \geqq \frac{1}{2}$ two distribution functions, $\iota = \iota_\nu = \iota_\nu(x)$ and $\kappa = \kappa_\nu = \kappa_\nu(x)$, such that*

$$\text{const. } t^{-\nu} J_\nu(t) = L(t; \iota_\nu) \tag{29}$$

and

$$\text{Const. } t^{-\nu} \int_0^1 r^{\nu+1} J_\nu(tr)\, dr = L(t; \kappa_\nu), \tag{30}$$

where the factors of proportionality are positive and are determined by the condition $L(0; \cdot) = 1$. The spectrum is always the interval $-1 \leqq x \leqq 1$.

Remark. $\int_0^1$ may be replaced by $\int_0^R$, where $0 < R < +\infty$. In fact, this change replaces κ_ν by a distribution function which is similar to κ_ν in the sense of § 1.

Proof. Suppose first only $\nu > -\frac{1}{2}$ and put $\iota_\nu'(x)$ proportional to $(1 - x^2)^{\nu - 1/2}$, if $-1 < x < 1$, and $\iota_\nu'(x) = 0$, if $|x| \geqq 1$. Then (29) is, in view of the integral definition of the Bessel functions, satisfied by the symmetric distribution function ι_ν. Furthermore, ι_ν is a convex distribution function if and only if $\iota_\nu'(|x|)$ is monotone non-increasing, i. e., if and only if $\nu \geqq \frac{1}{2}$. It also follows that the left-hand side of (30) is

$$\int_0^1 \int_0^1 r^{2\nu+1}(1-s^2)^{\nu-1/2} \cos(trs)\,dr\,ds$$

up to a constant factor which is positive by the definition of $\iota_\nu'(x)$. On placing $rs = x$, where s is fixed, the last integral may be written in the form

$$\int_0^1 s^{-2\nu-2}(1-s^2)^{\nu-1/2}\{\int_0^s x^{2\nu+1} \cos(tx)\,dx\}ds.$$

On partial integration this becomes

$$\int_0^1 F_\nu(x) x^{2\nu+1} \cos(tx)\,dx,$$

i. e.,

$$\int_{-1}^1 \tfrac{1}{2} F_\nu(|x|)\,|x|^{2\nu+1} e^{itx}dx,$$

where

(31) $$F_\nu(x) = \int_x^1 s^{-2\nu-2}(1-s^2)^{\nu-1/2}ds;\ 0 < x < 1,$$

the infinity of $F_\nu(x)$ at $x = 0$ being compensated by the factor $x^{2\nu+1}$. Hence a distribution function $\kappa_\nu(x)$ satisfying (30) may be obtained by choosing $\kappa_\nu'(x)$ proportional to the positive function $F_\nu(|x|)\,|x|^{2\nu+1}$ or equal to zero according as $|x| < 1$ or $|x| \geqq 1$. Thus $\kappa_\nu(x)$ is a convex distribution function if and only if $F_\nu(x)x^{2\nu+1}$ is monotone non-increasing in the interval $0 \leqq x \leqq 1$. Hence it is sufficient to prove that

(32) $$\{F_\nu(x)x^{2\nu+1}\}' < 0, \text{ where } 0 < x < 1 \text{ and } ' = d/dx.$$

Now let $\nu \geqq \frac{1}{2}$. Then, from (31),

$$F_\nu(x) \leqq (1-x^2)^{\nu-1/2} \int_x^1 s^{-2\nu-2}ds,\ 0 < x < 1,$$

since $\nu - \frac{1}{2} \geqq 0$; and

$$(2\nu+1)\int_x^1 s^{-2\nu-2}ds = x^{-2\nu-1} - 1 < x^{-2\nu-1},$$

so that

(33) $$F_\nu(x) < (2\nu+1)^{-1}x^{-2\nu-1}(1-x^2)^{\nu-1/2},\ 0 < x < 1.$$

On calculating the derivative of the product $F_\nu(x)x^{2\nu+1}$ by using the definition (31) of $F_\nu(x)$, it is easily found that (33) may be written in the form (32). This completes the proof of Theorem X.

Since from (29)

(34) $$L(t;\iota_{1/2}) = 2^{1/2}\Gamma(\tfrac{1}{2}+1)t^{-1/2}J_{1/2}(t) = t^{-1}\sin t,$$

hence, by Vieta's product for $t^{-1}\sin t$,

$$L(t;\iota_{1/2})=\prod_{j=1}^{\infty}\cos(t/2^j),$$

it is easy to see that

$$\prod_{k=1}^{\infty}\cos(t/k)=L(2t;\iota_{1/2})L(2t/3;\iota_{1/2})L(2t/5;\iota_{1/2})\cdots.$$

Hence it is clear from the product rules (4) and (5) that the distribution function τ defined by (19) is

$$\iota_{1/2}(x/2)*\iota_{1/2}(3x/2)*\iota_{1/2}(5x/2)*\cdots, \tag{35}$$

and is therefore convex in view of the convexity of $\iota_{1/2}$. This suggests more general

Remarks on the convexity of the projections of spherical equidistributions. Let

$$b_1>0,\ b_2>0,\cdots \text{ and } b_1^2+b_2^2+\cdots<+\infty.$$

Then it is clear from (29) and (30) that both infinite products

$$\prod_{m=1}^{\infty}L(b_mt;\iota_\nu),\qquad \prod_{m=1}^{\infty}L(b_mt;\kappa_\nu)$$

are uniformly convergent in every fixed finite interval $|t|<$ Const. It therefore follows from the rule (5) that

$$\mathrm{I}_\nu(x)=\iota_\nu(x/b_1)*\iota_\nu(x/b_2)*\cdots \tag{36}$$

and

$$\mathrm{K}_\nu(x)=\kappa_\nu(x/b_1)*\kappa_\nu(x/b_2)*\cdots \tag{37}$$

are convergent infinite convolutions. Since the convexity is preserved by a similarly transformation, by the convolution process and by a limit process (cf. § 1), Theorem X implies that I_ν and K_ν are convex distribution functions for any $\{b_m\}$. This holds for every $\nu\geqq\frac{1}{2}$.

If $\nu=\frac{1}{2}n-1$, where n is an integer, the convexity of the distribution function (36) may be interpreted as a result on a random walk problem in an n-dimensional space, where $n\geqq 3$ in view of $\nu\geqq\frac{1}{2}$. In fact, the convexity of a distribution function means that the density of probability is a non-increasing function of the distance $|x|$ from the "median" $x=0$ (cf. § 3 below). Now $\iota_\nu(x/b)$ is the distribution function obtained by projection from the equidistribution on the sphere $\xi_1^2+\cdots+\xi_n^2=b^2$, where $n=2\nu+2$; cf.

Jessen and Wintner [12], § 3 and § 5, where further references are given. It is interesting that the infinite convolution (36) in the case of the dimension number $n=2$, which belongs to $\nu=0<\frac{1}{2}$ and occurs in connection with almost periodic functions with linearly independent frequencies, is from the point of view of convexity just as exceptional as in the degenerate case $n=1$; the latter belongs to $\nu=-\frac{1}{2}$, hence, in view of

$$(38)\qquad L(t;\iota_{-1/2})=2^{-1/2}\Gamma(-\tfrac{1}{2}+1)t^{1/2}J_{-1/2}(t)=\cos t,$$

to symmetric Bernoulli convolutions.

The convex distribution function K_ν defined by (37), where $\nu\geqq-\frac{1}{2}$, admits in the case $\nu=\frac{1}{2}n-1$ of an interpretation similar to that of (36). In fact, $\kappa_\nu(x/b)$ is the distribution function obtained by projection from the equidistribution within the spherical volume $\xi_1^2+\cdot\cdot\cdot+\xi_n^2\leqq b^2$, where $n=2\nu+2$. For the Fourier transform L of the distribution function of the projection is easily found to be

$$\text{Const.}\int_0^b r^{n-1}(tr)^{1-n/2}J_{-1+n/2}(tr)\,dr,$$

an integral which is identical with $L(bt;\kappa_{-1+n/2})$ in view of (30). While it is clear from (38) and from the proof of Theorem X that if $n=1$ and $n=2$, then ι_ν is not convex, hence (36) need not be convex, the distribution function (37) is always convex in these cases also. In other words, $\kappa_{-1/2}$ and κ_0 are convex distribution functions. As far as $\nu=-\frac{1}{2}$, i. e., $n=1$ is concerned, the truth of the statement is easily verified from (38), (34) and (30) and also is geometrically clear. If $n=2$, then $\nu=0>-\frac{1}{2}$, and so the representation of $\kappa_\nu=\kappa_0$ in terms of (31) is valid. Hence it is sufficient to show that (32) holds for $\nu=0$. Now from (31)

$$\{xF_0(x)\}'=\int_x^1 s^{-2}(1-s^2)^{-1/2}ds-x^{-1}(1-x^2)^{-1/2}.$$

Hence

$$\{xF_0(x)\}''=-(1-x^2)^{-3/2}<0;\quad 0<x<1.$$

Thus $\{xF_0(x)\}'$ is steadily decreasing and will, therefore, satisfy the condition (32), where $\nu=0$, if it is negative in the neighborhood of $x=+0$. Now if $x\to+0$, then, by l'Hospital's rule,

$$\{xF_0(x)\}'\sim x\{xF_0(x)\}''=-x(1-x^2)^{-3/2}<0.$$

Thus the lowest dimension numbers, $n=1$ and $n=2$, are or are not exceptional with regard to convexity according as one considers the projection

of the equidistribution belonging to the boundary or to the interior of an n-dimensional sphere.

3. *On a postulate of Gauss.* Let Q' and Q'' be two places, Q a collinear place between them, d_0', d_0'', d_0 the "true" values and d', d'', d some measured values of the distances $Q'Q$, QQ'', $Q'Q''$ respectively. Instead of distances, one may think of masses or of arbitrary additive observables which are characterized by a real number. Suppose that the three "distances" have been measured very often. The result of a measurement will be, of course, almost never precisely the same as the result of a previous measurement, so that one has practically continuous series $d'-d_0'$, $d''-d_0''$, $d-d_0$ of errors of observation. If these series belong to reliable observations and have been obtained for practically every position of the intermediary point Q, the distribution of errors which are introduced by the limited accuracy of any measurement will satisfy three postulates of Gauss. The first of these conditions is satisfied for the normal law of Gauss (cf. Bessel [1]); Gauss [6] states the two others as general postulates. The three conditions in question will be conceded to be necessary for the case of "reliable" measurements, measurements where nothing is wrong with the instruments or with the observers.

(i) The three distribution curves plotted for the errors $d'-d_0'$, $d''-d_0''$ and $(d'+d'')-(d_0'+d_0'')$ must be similar to the distribution curve plotted for the error $d-d_0$ of the direct measurement of the distance $Q'Q''$, no matter what is the position of Q between Q' and Q''. The similarity of two distribution curves means, as in § 1, that they become identical by a suitable change of the unit of one of the two errors. As to this condition (i), cf. Bessel [1], Pólya [17].

(ii) The error $d-d_0$ must be just as probable and/or frequent as the error $-(d-d_0)$. In other words, the probability of an error must not depend on its sign but only on its magnitude.

(iii) If d_1 and d_2 are measured values of d_0 and $|d_1-d_0|<|d_2-d_0|$, the probability of the error d_1-d_0 must not be less than that of the error d_2-d_0. In other words, small errors must be at least as probable as large errors.

Conditions (ii) and (iii) together may be expressed by saying that the density of probability must be a non-increasing function of the absolute deviation from the "true" value. The latter was so far undefined but may now be defined as the median.

Postulates (i) and (ii) are independent of each other, since the asym-

metric arc tan-distributions of Cauchy [4] satisfy (i) without satisfying (ii). The question of whether all three postulates are independent of each other has apparently not been treated in the literature. In what follows, it will be shown that (iii) *is a consequence of* (i) *and* (ii). Conditions (i), (ii) and (iii) are, in the terminology of § 1, the conditions of stability, symmetry and convexity. Hence what one has to prove is

THEOREM XI. *Every symmetric stable distribution function is a convex distribution function.*

Remark. Since the symmetric distribution functions ω_λ defined by (27) satisfy the stability criterion (6), Theorem XI implies that every ω_λ is convex, as stated by Theorem VIII. On the other hand, the statement of Theorem XI, viz. that the postulates (i) and (ii) imply (iii), does not sound like an analytical statement but rather like one which, when true, must be provable without any analytical tools; hence the convexity of every ω_λ does not seem to be an analytical fact. While an approach to Theorem VIII from this abstract direction is perhaps not impossible, the way to be followed will be that of deducing Theorem XI from Theorem VIII. To this end it will be necessary to know that every symmetric stable distribution function is, in the main, an ω_λ. Cauchy [4], p. 101 and Lévy [14], p. 254, derived by summary considerations an exponential representation of the Fourier transform of an arbitrary stable distribution function, and this exponential representation shows that Theorem XI is implied by Theorem VIII. However, for a complete proof of Theorem XI a detailed proof of the exponential representation in question will be needed. This proof will also save a similar consideration in § 4. The proof of Theorem XI will occupy this whole § 3.

In order not to interrupt the proof in what follows, let it here be observed that if $l(s)$, where $0 < s < +\infty$, is a non-vanishing real or complex function such that for every $s > 0$, for every $v > 0$ and for some real constant $\lambda > 0$

$$(39) \qquad l(s^{1/\lambda}v) = [l(v)]^s, \text{ then } l(s) = \exp(-Cs^\lambda),$$

where C is a constant which need not be real. This statement is easily verified by choosing $v = 1$ and denoting by C one of the logarithms of $[l(1)]^{-1}$.

Now let σ be an arbitrary stable distribution function which is distinct from the unit distribution function χ, so that the positive number $c = c(a, b)$ defined at the end of § 1 is unique. Put $c_{m+1} = c(1, c_m)$, where $c_1 = 1$ and $m = 1, 2, \cdots$. Then

$$(40) \qquad [L(t;\sigma)]^m = L(c_m t;\sigma), \qquad (m = 1, 2, \cdots),$$

in view of (6). The positive sequence $\{c_m\}$ will be considered as undefined if $\sigma = \chi$, in which case (40) is satisfied by any c_m, since $L(t;\chi) = 1$ for every t.

It is easy to prove that

$$(41) \qquad c_2 > c_1 = 1 \text{ and } \limsup_{m=\infty} c_m = +\infty.$$

First, on choosing in (40) the exponent $m = 2$ and iterating the resulting relation k times, it is seen that the 2^k-th power of $L(t;\sigma)$ is $L(c_2{}^k t;\sigma)$. Since there exists for every m but one c_m, it follows from (40) that $c_m = c_2{}^k$ where $m = 2^k$ and $k = 1, 2, \cdots$. Hence if the first of the statements (41), viz. $c_2 > 1$, is true, then so is the second. Suppose, if possible, that $c_2 > 1$ is false. If $c_2 = 1$, then $L(t;\sigma)$ is, in view of (40), its own square for every t, since $c_1 = 1$. Hence $L(t;\sigma)$ is either 0 or 1 for a given t. Since $L(t;\sigma)$ is a continuous function, it follows that either $L(t;\sigma) = 0$ for every t or $L(t;\sigma) = 1$ for every t. The first case is impossible, since $L(0;\sigma) = 1$ for every σ, and the second case is impossible, since $\sigma \neq \chi$ by supposition. Now let $c_2 < 1$. Then $L(c_2{}^k t;\sigma)$ tends, as $k \to +\infty$, to 1 for every t, since $L(0;\sigma) = 1$. Hence the 2^k-th power of $L(t;\sigma)$ is 1 for every t, which again implies the contradiction $\sigma = \chi$. This completes the proof of (41).

It follows that

$$(42) \qquad L(t;\sigma) \neq 0, \quad -\infty < t < +\infty,$$

for every stable σ. This is clear if $\sigma = \chi$. Let $\sigma \neq \chi$ and suppose, if possible, that $L(t_0;\sigma) = 0$ for some $t = t_0$. Then it is clear from (40) that $L(t;\sigma)$ vanishes at $t = t_0/c_m$ also. Hence it is seen from the second of the relations (41) that $t = 0$ is either a zero or a cluster point of zeros of $L(t;\sigma)$. Both cases are impossible, since $L(t;\sigma)$ is a continuous function and $L(0;\sigma) = 1$. This proves (42).

Thus there exists for $-\infty < t < +\infty$ a unique continuous $\log L(t;\sigma)$ which vanishes at $t = 0$. In what follows, $[L(t;\sigma)]^s$ will mean $\exp\{s \log L(t;\sigma)\}$ with the previous definition of the logarithm. Thus $[L(t;\sigma)]^s$ is a continuous function of t and s together.

Let m and n be arbitrary positive integers. Repeated application of (40) shows that

$$[L(tc_m/c_n;\sigma)]^n = L(tc_m;\sigma) = [L(t;\sigma)]^m.$$

Hence, on using the above definition of the power $[L(t;\sigma)]^s$,

$$(43) \qquad L(tc_m/c_n;\sigma) = [L(t;\sigma)]^{m/n}.$$

This implies that the Fourier transform of the distribution function $\sigma(ax)$,

where $a = c_m/c_n$, only depends on m/n. Since, as pointed out in § 1, the relation $\sigma(a_1x) = \sigma(a_2x)$ is impossible for $a_1 \neq a_2$ unless $\sigma = \chi$, and since $\sigma \neq \chi$ by assumption, it follows that c_n/c_m depends only on m/n.

Let s be an arbitrary positive number, $s_1 = n_1/m_1, \cdots, s_j = n_j/m_j, \cdots$ a sequence of rational positive numbers which tend to s as $j \to \infty$, and let $\sigma_j(x)$ denote the distribution function defined by

$$\sigma_j(x) = \sigma(xc_{m_j}/c_{n_j}),$$

so that

$$L(t;\sigma_j) = [L(t;\sigma)]^{s_j}$$

in view of (43). Thus, since $[L(t;\sigma)]^s$ is a continuous function of t and s together, it follows from the continuity theorem (§ 1) and from the assumption $s_j \to s$, where $j \to \infty$, that there exists a distribution function $\tau = \tau_s$ such that $\sigma_j \to \tau_s$ and

$$(44) \qquad L(t;\tau_s) = [L(t;\sigma)]^s.$$

The last relation shows that $\tau_s(x)$ is uniquely determined by s, i. e. that $\tau_s(x)$ is independent of the sequence $\{s_j\}$ of rational numbers by which s has been approximated. Since $\sigma_j \to \tau_s$,

$$\sigma(xc_{m_j}/c_{n_j}) \to \tau_s(x), \ j \to \infty,$$

by the definition of σ_j. Hence it is clear from $\sigma \neq \chi$ that c_{n_j}/c_{m_j} tends to a finite positive limit $f = f(s)$ for which

$$\sigma(x/f(s)) = \tau_s(x);$$

since $\tau_s(x)$ depends only on s, the same holds for the limit $f(s)$ of c_{n_j}/c_{m_j} in view of $\sigma \neq \chi$. Now the last representation of τ_s is equivalent to

$$L(f(s)t;\sigma) = L(t;\tau_s).$$

Consequently,

$$(45) \qquad L(f(s)t;\sigma) = [L(t;\sigma)]^s$$

in view of (44). On using the continuity theorem (§ 1) and the assumption $\sigma \neq \chi$ once more, it is seen from (45) that the function $f(s)$, which is positive by its definition, is a continuous function of s, where $0 < s < +\infty$. Suppose that $f(s') = f(s'')$ for some pair of positive numbers s', s''. Then it is clear from (45) and (42) that either $s' - s'' = 0$ or $L(t;\sigma) = 1$ for every t. Since the second possibility is excluded by the assumption $\sigma \neq \chi$, it follows that $f(s') = f(s'')$ only when $s' = s''$. Consequently, $f(s)$ is a strictly monotone

function in view of its continuity. Since $c_m = f(m)$ in virtue of (40) and (45), it is clear from (41) that $f(s)$ is monotone increasing and $f(+\infty) = +\infty$.

The number c_m has so far been defined only for $m = 1, 2, \cdots$. If $s > 0$ is not an integer, put $c_s = c_m$, where m is the least integer exceeding s. Since $f(m/n) = c_m/c_n$ in view of (43) and (45), and since $f(s)$, where $0 < s < +\infty$, is a positive monotone continuous function, it is clear that if $s > 0$ is fixed and k is an integer which tends to $+\infty$, then $c_{sk}/c_k \to f(s)$, where sk is the product of s and k. Similarly, $c_{srk}/c_{rk} \to f(s)$, if $s > 0$ and $r > 0$ are fixed. Since c_{srk}/c_k is the product of c_{srk}/c_{rk} and c_{rk}/c_k, it follows by letting $k \to +\infty$ that $f(s)$ satisfies the functional equation $f(sr) = f(s)f(r)$. Hence, since $f(s)$ is positive and continuous, $f(s) = s^{\kappa}$, where κ is a real constant and $0 < s < +\infty$. Furthermore, $\kappa > 0$ in virtue of $f(+\infty) = +\infty$. On placing $\lambda = 1/\kappa$, it follows from (45) that

$$L(s^{1/\lambda}t;\sigma) = [L(t;\sigma)]^s, \qquad (s > 0,\ t \gtreqless 0), \tag{46}$$

where σ is an arbitrary stable distribution function and λ a positive constant depending on σ. If $\sigma = \chi$, then the relation (46) is but $1 = 1$, so that $\sigma = \chi$ need not be excluded. On choosing $t > 0$ and placing $t = v$, it is seen from (46) and (39) that there exists a constant C for which

$$L(t;\sigma) = \exp(-Ct^{\lambda}), \text{ if } 0 < t < +\infty. \tag{46a}$$

This holds for $t = 0$ also, since $L(0;\sigma) = 1$. Now $L(t;\sigma)$ and $L(-t;\sigma)$ are conjugated complex in view of (1). Hence

$$L(t;\sigma) = \exp(-\bar{C}\,|\,t\,|^{\lambda}), \text{ if } -\infty < t < 0. \tag{46b}$$

Now suppose that the stable distribution function σ is symmetric, i. e., that $\sigma(x) = 1 - \sigma(-x)$. Then $L(t;\sigma)$ is a real function in virtue of (1); hence $C = \bar{C}$, and so

$$L(t;\sigma) = \exp(-C\,|\,t\,|^{\lambda}), \quad -\infty < t < +\infty,$$

where $\lambda > 0$. Furthermore, $C < 0$ is impossible, since $|\,L(t;\sigma)\,| \leqq 1$ for every σ and for every t. Thus either $C = 0$ or $C > 0$. If $C = 0$, then $L(t;\sigma) = 1$ for every t, hence $\sigma = \chi$. If $C > 0$, on replacing $\sigma(x)$ by $\sigma(C^{-1/\lambda}x)$ it may be assumed that $C = 1$. Hence if σ is symmetric and stable, then either $\sigma = \chi$ or σ is similar to a distribution function ω_{λ} which satisfies (27). Now if $\sigma(x)$ is a convex distribution function, then so is $\sigma(ax)$ for every $a > 0$. Furthermore, χ is a convex distribution function; cf. § 1. Hence Theorem XI is implied by Theorem VIII in view of the Remark which follows Theorem VIII.

4. *The multidimensional case.* Let $y = (y_1, \cdots, y_n)$ be a point of the real n-dimensional Cartesian space R_y. By a distribution function in R_y is meant an absolutely additive monotone set-function $\phi = \phi(E)$ which is defined for every Borel set E of R_y and is such that $\phi(R_y) = 1$. If the dimension number n of R_y is 1 and F_x denotes the Borel set $-\infty < y < x$, then $\sigma(x) = \phi(F_x)$ is a distribution function in the sense of § 1. Correspondingly, the Borel set E is termed in the case of an arbitrary n a continuity set of ϕ if $\phi(E_c) = \phi(E_i)$, where E_c denotes the closure of E and E_i the set of the interior points of E_i, hence possibly the empty set, in which case $\phi(E_i) = 0$. If ϕ is fixed, a set E is a continuity set of ϕ "in general"; namely in the same sense as a monotone function of a single variable is continuous "in general," that is to say up to a set of points which is at most enumerable. Correspondingly, a sequence $\{\phi_m\}$ of distribution functions in R_y is said to be convergent if there exists a distribution function ϕ such that $\phi_m(E) \rightarrow \phi(E)$ holds for every continuity set E of ϕ. The distribution function $\phi * \psi$ in R_y which represents the convolution of ϕ and ψ is defined in accordance with the case $n = 1$ of a single dimension (§ 1) and has the same properties as in the case $n = 1$. In particular, on placing

$$\Lambda(u; \phi) = \int_{Ry} e^{iuy} \phi(dR_y), \tag{47}$$

where the integral is a Radon integral, $u = (u_1, \cdots, u_n)$ is a point in a real space R_u and uy is the scalar product $u_1y_1 + \cdots + u_ny_n$, one has

$$\Lambda(u; \phi * \psi) = \Lambda(u; \phi)\Lambda(u; \psi) \tag{48}$$

for every vector u in R_u. Furthermore, $\{\phi_m\}$ is a convergent sequence of distribution function if and only if the sequence $\{\Lambda(u; \phi_m)\}$ of the Fourier transforms is uniformly convergent within every fixed sphere of the space R_u, and $\lim \Lambda(u; \phi_m)$ is then $\Lambda(u; \lim \phi_m)$ for every u. This again will be referred to as the continuity theorem. A detailed treatment of the theory of the distribution functions in R_y is given by Haviland [9], [10], where further references may be found. The continuity theorem, when applied to the sequence $\phi, \psi, \phi, \psi, \phi, \cdots$, implies that there belongs to every Fourier transform $\Lambda(u; \phi)$ but one ϕ. For an explicit inversion formula cf. Haviland [10].

The distribution function ϕ is called absolutely continuous if there exists in R_y a measurable function $\delta(y) = \delta(y_1, \cdots, y_n)$ such that

$$\phi(E) = \int_E \delta(y)\mu(dR_y) \tag{49}$$

for every Borel set E, where $\mu(E)$ is the set-function representing the ordinary Lebesgue measure in R_y, so that (49) is an ordinary Lebesgue integral. The function $\delta(y)$, if it exists, is termed the density of $\phi(E)$ and is undetermined on a set of measure zero, a remark which will not be repeated. Since $\phi(E)$ is monotone, $\delta(y) \geqq 0$. The inversion formula mentioned before implies that if $|\Lambda(u;\phi)|$ has a finite integral over R_u, then ϕ is absolutely continuous with a uniformly continuous and bounded density $\delta(y)$ which may be obtained from (47) for every y by the ordinary Fourier inversion,

$$(50) \qquad \delta(y) = (2\pi)^{-n} \int_{R_u} e^{-iyu} \Lambda(u;\phi) \mu(dR_u).$$

Let O denote the origin of R_y when this single point is considered as a Borel set. Thus if E is an arbitrary Borel set, $\phi(EO) = \phi(O) \geqq 0$ or $\phi(EO) = 0$ according as the point O is or is not in E, and $\phi(O) = 0$ if and only if the point O is a continuity set of ϕ.

Let $\phi_\Omega = \phi_\Omega(E)$ denote the distribution function $\phi(\Omega E)$, where Ω is a fixed orthogonal matrix representing a rotation of R_y about O and ΩE denotes the Borel set into which E is turned by this transformation of R_y into itself. Since the scalar product uy is invariant under rotations, it is clear from (47) that $\Lambda(u;\phi_\Omega) = \Lambda(\Omega^{-1}u;\phi)$. In particular, $\Lambda(u;\phi_\Omega)$ is independent of Ω if and only if $\Lambda(u;\phi)$ is a function of the length $|u|$ alone. Hence $\Lambda(u;\phi)$ is a function of $|u|$ alone if and only if $\phi_\Omega = \phi$ for every rotation Ω. If this condition is satisfied, ϕ is said to be of radial symmetry. It is clear from (49) that an absolutely continuous ϕ is of radial symmetry if and only if its density $\delta(y)$ is a function of $|y|$ alone. If in addition $|\Lambda(u;\phi)| = |\Lambda(|u|;\phi)|$ has a finite integral over R_u, then

$$\delta(y) = B_n \int_0^{+\infty} \left\{ \int_0^{\pi} \exp(-i\,|y|\,|u|\cos\vartheta) \sin^{n-2}\vartheta\, d\vartheta \right\} \times \Lambda(|u|;\phi) n\, |u|^{n-1} d\,|u|, \quad B_n > 0,$$

as seen from (50) by introducing polar coördinates, ϑ being the angle between the vectors y and u. This tacitly assumes that $n > 1$. However, on expressing the inner integral in terms of Bessel functions and placing $|u| = s$, one obtains

$$(51) \quad \delta(y) = \delta(|y|) = A_n\, |y|^{1-n/2} \int_0^{+\infty} s^{n/2} J_{-1+n/2}(|y|\, s) \Lambda(s;\phi)\, ds, \quad A_n > 0,$$

a formula which holds, in view of (38) and (2), for $n = 1$ also; for if $n = 1$, then radial symmetry means symmetry in the sense of § 1. The Fourier inversion of (51) is the Cauchy-Poisson formula for spherical waves.

Let $|y|$ denote the length of the vector y. If there exists in the interval $0 < r < +\infty$ a monotone non-increasing function $\eta(r)$ such that

$$(52) \qquad \phi(E) = \phi(EO) + \int_E \eta(|y|)\mu(dR_y)$$

for every Borel set E, then $\phi(E)$ will be said to be convex. This definition is justified by the fact that if $n = 1$, then $\phi(E)$ is convex if and only if $\sigma(x) = \phi(F_x)$ is a convex distribution function in the sense of § 1, the Borel set F_x being the interval $-\infty < y < x$. It is clear that every convex $\phi(E)$ is of radial symmetry. Also

$$(53) \qquad \phi(E_r) = \phi(O) + \alpha_n \int_0^r \eta(s)s^{n-1}ds,$$

where E_r denotes the sphere $|y| < r$ and α_n a positive constant which depends only on the dimension number n. It will always be supposed that $r > 0$. On comparing (52), (53) with (49) it is seen that a convex ϕ is absolutely continuous if and only if $\phi(O) = 0$, in which case its density $\delta(y)$ is the non-increasing function $\eta(|y|) \geqq 0$. Hence it is clear that if ϕ is convex, there exist convex $\phi_1, \phi_2, \cdots$ such that on the one hand $\phi_m \to \phi$ and on the other hand every ϕ_m has a density $\delta_m(y) = \delta_m(r)$ which possesses a continuous derivative in the interval $0 < r < +\infty$.

THEOREM XII. *If $\phi_1, \phi_2, \cdots$ are convex and $\phi_m \to \phi$, then ϕ also is convex.*

Proof. Since ϕ_m is convex, hence of radial symmetry, $\phi = \lim \phi_m$ clearly is of radial symmetry. Consequently, (52) is implied by (53), and so it is sufficient to prove that condition (53) is satisfied by a non-negative non-increasing η. Since ϕ_m is convex,

$$(54) \qquad \phi_m(E_r) = \phi_m(O) + \alpha_n \int_0^r \eta_m(s)s^{n-1}ds,$$

where η_m is non-negative and non-increasing. Thus, since $0 \leqq \phi_m \leqq 1$,

$$1 \geqq \alpha_n \int_0^r \eta_m(s)s^{n-1}ds \geqq \alpha_n \int_\epsilon^r \eta_m(s)s^{n-1}ds,$$

where $\epsilon > 0$ is arbitrarily fixed and $\epsilon \leqq r < +\infty$. The integrand is the product of two monotone functions which increase, if $n > 1$, in opposite directions. It is, however, seen from the last inequality that

$$\alpha_n^{-1} \geqq \eta_m(2\epsilon) \int_\epsilon^{2\epsilon} s^{n-1}ds.$$

Hence the sequence $\{\eta_m(2\epsilon)\}$ is bounded, implying that $\{\eta_m(r)\}$ is uniformly bounded for $2\epsilon \leqq r < +\infty$, where $\epsilon > 0$ is arbitrarily small. Thus it follows from Helly's compactness theorem that the sequence $\{\eta_m(r)\}$ contains a subsequence which is convergent in the whole range $0 < r < +\infty$. The corresponding subsequence of $\{\eta_m(r)r^{n-1}\}$ is uniformly bounded in every finite interval $0 < a \leqq r \leqq b$ and may, therefore, be integrated term by term. Consequently, since $\phi_m \rightarrow \phi$, it is seen from (54) that

$$\phi(E_b) - \phi(E_a) = \alpha_n \int_a^b \eta(s)s^{n-1}ds, \tag{55}$$

where η denotes the limit of the selected subsequence of $\{\eta_m\}$ and is, therefore, a non-negative and non-increasing function of $r > 0$. Since $a > 0$ and $b > 0$ are arbitrary, (55) clearly implies (53). This completes the proof of Theorem XII. Incidentally, a standard argument shows that the selection is, in reality, superfluous, since $\eta_m(r) \rightarrow \eta(r)$ at every continuity point r of $\eta(r)$. Of course, $\phi_m \rightarrow \phi$ does not imply $\phi_m(O) \rightarrow \phi(O)$, since $E = O$ need not be a continuity set of $\phi(E)$.

THEOREM XIII. *If $\phi_1(E)$ and $\phi_2(E)$ are convex, then so is $\phi_1(E) * \phi_2(E)$.*

Proof. It is clear from (48) and from the continuity theorem of the Fourier transform (47) that $\phi_{1m} \rightarrow \phi_1$ and $\phi_{2m} \rightarrow \phi_2$ imply $\phi_{1m} * \phi_{2m} \rightarrow \phi_1 * \phi_2$. Hence, by the remark which precedes Theorem XII, it is sufficient to prove Theorem XIII under the restriction that $\phi_i(E)$, where $i = 1$ and $i = 2$, is absolutely continuous and has a density $\delta_i(y) = \delta_i(|\, y \,|) = \delta_i(r)$ which admits of a continuous derivative for $0 < r < +\infty$. Since $\delta_i(r)$ is non-increasing, $\delta_i'(r) \leqq 0$, while, of course, $\delta_i(r) \geqq 0$. Since $\phi_i(E)$ is of radial symmetry, $\Lambda(u; \phi_i)$ depends only on $|\, u \,|$. Hence $\Lambda(u; \phi_1 * \phi_2)$, as product of $\Lambda(u; \phi_1)$ and $\Lambda(u; \phi_2)$, is a function of $|\, u \,|$ alone. Thus $\phi_1 * \phi_2$ is of radial symmetry. Since ϕ_i is absolutely continuous, so is $\phi_1 * \phi_2$ (cf., e. g., Kershner and Wintner [13], p. 543, footnote, where $n = 1$ but the proof holds for any n). Let $\delta(y) = \delta(|\, y \,|) = \delta(r)$ denote the density of $\phi_1 * \phi_2$. Then, by the definition of a convolution (cf. Haviland [9]),

$$\delta(|\, y \,|) = \delta(y) = \int_{Rv} \delta_1(y - v)\delta_2(v)\mu(dR_v),$$

where $y - v$ is the difference of the vectors y and v. Since the functions $\delta_i(r)$, $\delta_i'(r)$ are of constant sign, it is clear that $\delta'(r)$ exists and may be obtained by formal differentiation,

$$\delta'(r) = \delta'(|\,y\,|) = \int_{R_v} \delta_1'(y - v)\delta_2(v)\mu(dR_v),$$

the differentiation of $\delta_1(y - v) = \delta_1(|\,y - v\,|)$ being one with respect to the radius vector. Let H_v and $R_v - H_v$ be the two halves of R_v into which R_v is cut by a plane through the origin of R_v, and let H_v be chosen such that it contains the arbitrarily fixed point $y \neq (0, \cdot\cdot\cdot, 0)$. On writing $y - v$ instead of v as integration variable, it is seen from $\delta_i(y) = \delta_i(|\,y\,|) = \delta_i(-y)$ and from the directional character of the differentiation that

$$\delta'(r) = \delta'(|\,y\,|) = \int_{H_v} \delta_1'(|\,v\,|)\{\delta_2(|\,y - v\,|) - \delta_2(|\,y + v\,|)\}\mu(dR_v).$$

For if v is in H_v, then $-v$ is in $R_v - H_v$. Now y is in H_v. Hence y lies nearer to every point v of H_v than to the corresponding point $-v$ of $R_v - H_v$. Thus $|\,y - v\,| < |\,y + v\,|$ for every v in H_v. Since $\delta_2'(r) \leqq 0$, it follows that $\delta_2(|\,y - v\,|) \geqq \delta_2(|\,y + v\,|)$ for every v in H_v, i. e., that the difference $\{\ \}$ in the last integral is non-negative in the whole domain of integration. The factor $\delta_1'(|\,v\,|)$ of the integrand is nowhere positive. Hence $\delta'(r) \leqq 0$, where r is arbitrary. Consequently, $\delta(r)$ is monotone non-increasing. Since $\phi_1 * \phi_2$ is absolutely continuous, hence vanishes for $E = O$, the proof of Theorem XIII is herewith completed in view of (52), (53) and (49).

If $u \neq (0, \cdot\cdot\cdot, 0)$, put $e_u = u/|\,u\,|$, so that e_u is the unit vector which has the same direction as u. The next Theorem contains an extension of a fact mentioned in the Remark which follows Theorem VIII.

THEOREM XIV. *If a distribution function ϕ is such that, for some constant $\lambda > 0$ and for some function $C = C(e_u)$ of the direction,*

$$\Lambda(u;\phi) = \exp\{-C(e_u)\,|\,u\,|^{\lambda}\}, \tag{56}$$

then either $\lambda \leqq 2$ or $C(e_u) = 0$ for every u; in the latter case ϕ is the unit distribution function χ.

Proof. It is seen from (47) that $|\,\Lambda(u;\phi)\,| \leqq 1$ for every u and $\Lambda(u;\phi) = 1$ for $u = (0, \cdot\cdot\cdot, 0)$. Since

$$|\,\exp\{-C(e_u)\}\,| = |\,\Lambda(e_u;\phi)\,| \leqq 1,$$

the real part of $C(e_u)$ is bounded from below. Hence, on taking the real part of

$$\int_{R_y} (1 - e^{iuy})\phi(dR_y) = 1 - \Lambda(u;\phi),$$

it is clear from (56) that

$$\int_{R_y}(1-\cos uy)\phi(dR_y)=O(|u|^{\lambda}) \text{ as } |u|\to 0.$$

Thus, since $\alpha^{-2}(1-\cos\alpha)$ has in the interval $-1\leqq\alpha\leqq 1$ a positive minimum, it follows from

$$0\leqq\int_{|u||y|\leqq 1}\leqq\int_{R_y}=O(|u|^{\lambda})$$

that

$$\int_{|u||y|\leqq 1}|y|^2\phi(dR_y)=O(|u|^{\lambda-2}) \text{ as } |u|\to 0, \tag{57}$$

the integration being extended over the sphere $|y|\leqq|u|^{-1}$ of R_y. This sphere tends to R_y as $|u|\to 0$. Now if $\lambda>2$, then $O(|u|^{\lambda-2})\to 0$ as $|u|\to 0$; hence, in view of (57),

$$\int_{R_y}|y|^2\phi(dR_y)=0,$$

which is possible only when the spectrum of ϕ consists of the origin of R_y, i. e., when $\phi=\chi$. This completes the proof of Theorem XIV.

The limiting case $\lambda=2$ may be characterized by means of the "standard deviation" (dispersion)

$$\mathrm{M}(\phi)=\int_{R_y}|y|^2\phi(dR_y)=\int_{R_y}\sum_{j=1}^{n}|y_j|^2\phi(dR_y)\leqq+\infty \tag{58}$$

as follows:

THEOREM XV. *A given distribution function ϕ satisfies* (56) *for $\lambda=2$ and for some function $C(e_u)$ of the direction if and only if $0\leqq\mathrm{M}(\phi)<+\infty$, where $\mathrm{M}(\phi)=0$ only when $\phi=\chi$.*

Proof. If $\lambda=2$, then $O(|u|^{\lambda-2})$ remains bounded as $|u|\to 0$, hence $\mathrm{M}(\phi)<+\infty$ in view of (57). Conversely, let $\mathrm{M}(\phi)<+\infty$. Then it is clear from (58) that the integrals

$$\mu_j=\int_{R_y}y_j\phi(dR_y),\ \mu_{pq}=\int_{R_y}y_py_q\phi(dR_y), \text{ where } j,p,q=1,2,\cdots,n, \tag{59}$$

are absolutely convergent in virtue of the Schwarz inequality. It follows therefore from (47) that $\Lambda(u;\phi)$ has at every point $u=(u_1,\cdots,u_n)$ of R_u continuous partial derivatives of the first and second order, and that these derivatives may be obtained from (47) by differentiation beneath the integral sign. Hence, on applying to $\Lambda(u;\phi)$ Taylor's formula in the neighborhood of $u=(0,\cdots,0)$,

$$\Lambda(u;\phi)=1+i\sum_{j=1}^{n}\mu_j u_j-\tfrac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}\mu_{pq}u_p u_q+o\,(\sum_{j=1}^{n}u_j^2)$$

as $|u|\to 0$. On the other hand,

$$\Lambda(u;\phi)=1-C(e_u)\{(\sum_{j=1}^{n}u_j^2)^{\lambda/2}+o\,((\sum_{j=1}^{n}u_j^2)^{\lambda/2})\},\ |u|\to 0,$$

in view of (56). Now suppose that $\phi\neq\chi$. Then, since $\Lambda(u;\phi)=1$ for every u only when $\phi=\chi$, the factor $C(e_u)$ occurring in the second approximate representation of $\Lambda(u;\phi)$ does not vanish identically. Furthermore, it is clear from (58) and (59) that $\phi\neq\chi$ implies $\mathrm{M}(\phi)>0$, hence $\mu_{pp}(\phi)>0$ for at least one p, so that the quadratic form occurring in the first approximate representation of $\Lambda(u;\phi)$ does not vanish identically. Consequently,

$$\sum_{j=1}^{n}\mu_j u_j=0\quad\text{and}\quad-\tfrac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}\mu_{pq}u_p u_q=C(e_u)(\sum_{j=1}^{n}u_j^2)^{\lambda/2},$$

for every u. Since the last relation implies $\lambda=2$, Theorem XV follows.

It also follows that the matrix $\|-\tfrac{1}{2}\mu_{pq}\|$ is $C(e_u)$ times the unit matrix. Consequently, since the integrals (59) are independent of u, hence of e_u, the function $C(e_u)$ also is independent of e_u. This leads to

THEOREM XVI. *If a distribution function ϕ satisfies* (56) *for $\lambda=2$ and for some function $C=C(e_u)$ of the direction, then C is a real non-negative constant.*

Proof. As just shown, $C(e_u)$ is independent of e_u. Hence

$$\Lambda(u;\phi)=\exp\,(-C\,|u|^2),$$

where C is a constant. Since $|\Lambda(u;\phi)|\leqq 1$ in view of (47), it follows by letting $|u|\to\infty$ that the real part of C is non-negative. This implies Theorem XVI, since $\Lambda(u;\phi)$ and $\Lambda(-u;\phi)$ are conjugated complex in view of (47).

Remark. Since (56) implies

$$\Lambda(u;\phi)\sim 1-C(e_u)|u|^{\lambda},\ \text{where}\ |u|\to 0,$$

it is obvious that $\Lambda(u;\phi)$ has at the single point $u=(0,\cdots,0)$ partial derivatives of order $[\lambda]$, where $[\lambda]$ denotes the least integer not exceeding λ. This fact may be extended in the usual way to the case of fractional differentiation, only that the order must then be replaced by $\lambda-\epsilon$. On the other hand, it is *not* obvious that the derivatives at $u=(0,\cdots,0)$ may be obtained by *formal* differentiation of the integral (47). In fact, it is not clear that

$$\int_{R_y} |y|^\nu \phi(dR_y) < +\infty$$

for $\nu = \lambda$ or for every positive $\nu < \lambda$. Otherwise Theorem XV would hardly need a proof. As far as this difficulty is concerned, there is no real difference between Lévy's case $n = 1$ and the case of an arbitrary dimension number n.

Theorem XVI has been stated for $n = 1$ by Lévy [14], p. 261, footnote (²).

If c is a positive number, let $\phi_c(E)$ denote the distribution function $\phi(cE)$, it being understood that cE is the set of all points $cy = (cy_1, \cdots, cy_n)$, where y is an arbitrary point of E. It is clear from (47) that $\Lambda(u; \phi_c) = \Lambda(u/c; \phi)$. As in the case $n = 1$ of § 1, and for the same reason, $\phi(E)$ will be said to be a stable distribution function if there exists for every $a > 0$ and every $b > 0$ a $c = c(a, b) > 0$ such that $\phi_a * \phi_b = \phi_c$, i. e.,

$$(60) \qquad \Lambda(u/a; \phi)\Lambda(u/b; \phi) = \Lambda(u/c; \phi).$$

THEOREM XVII. *A distribution function ϕ is stable if and only if $\Lambda(u; \phi)$ may be represented in the form* (56), *where the positive constant λ and the function $C(e_u)$ of the direction depend on ϕ.*

Remark. It is not stated, and it is not true, that if λ and $C(e_u)$ are given, (56) may be satisfied by a distribution function ϕ; Theorem XVII only decides when is a *given* ϕ a stable ϕ. For $n = 1$ cf. Lévy [14].

Proof. If (56) is satisfied, on placing

$$c = c(a, b) = (a^{-\lambda} + b^{-\lambda})^{-1/\lambda} > 0, \text{where } a > 0 \text{ and } b > 0,$$

(60) is clearly satisfied, and so ϕ is stable. Conversely, suppose that ϕ is stable. Then there exists a constant $\lambda > 0$ such that

$$\Lambda(s^{1/\lambda}u; \phi) = [\Lambda(u; \phi)]^s (\neq 0)$$

for every $s > 0$ and for every vector u. This is the extension of (46) for the case of an arbitrary dimension number n and is proven exactly in the same way as in the case $n = 1$ of § 3. Now, since $u = |u| e_u$,

$$\Lambda(s^{1/\lambda} | u | e_u; \phi) = [\Lambda(| u | e_u; \phi)]^s.$$

Hence on placing

$$v = | u | \text{ and } l(v) = \Lambda(ve_u; \phi),$$

where e_u is arbitrarily fixed, (56) follows from (39). This completes the proof of Theorem XVII.

The next Theorem might be of some physical interest in the case $n = 3$.

THEOREM XVIII. *If the dispersion* (58) *of a stable distribution function* ϕ *is neither zero nor infinite, then* ϕ *must be the Gauss-Maxwell law of radial symmetry.*

Proof. By Theorem XVII, $\Lambda(u;\phi)$ is of the form (56). Since $\mathrm{M}(\phi) < +\infty$ by assumption, $\lambda = 2$ and $C(e_u)$ is a real non-negative constant by Theorems XV and XVI. This constant cannot be zero, since otherwise $\Lambda(u;\phi) = 1$ for every u, i.e., $\phi = \chi$, which is excluded by the assumption $\mathrm{M}(\phi) > 0$. Thus $\Lambda(u;\phi) = \exp(-C\,|\,u\,|^2)$, where C is a positive constant. This implies Theorem XVIII, since $\exp(-r^2)$ is, up to constant factors, self-reciprocal under the Fourier cosine transformation.

The problem discussed at the beginning of § 3 may be extended to the multidimensional case. The postulate which corresponds to (iii) is again implied by the two other postulates, as shown by

THEOREM XIX. *Every stable distribution function of radial symmetry is convex.*

Proof. Let ϕ be stable. Then $\Lambda(u;\phi)$ is of the form (56) in view of Theorem XVII. Thus if ϕ is of radial symmetry, and so $\Lambda(u;\phi)$ a function of $|\,u\,|$ alone, then $C(e_u)$ is a constant. It is shown as in the Proof of Theorem XVI that this constant C is positive or zero. If $C = 1$, then $\Lambda(u;\phi) = \exp(-|\,u\,|^\lambda)$, where $0 < \lambda \leqq 2$ in view of Theorem XIV, and it will be shown by Theorem XX that this ϕ exists and is convex for $n = 1, 2, \cdots$ and $0 < \lambda \leqq 2$. The case $C > 0$ may be reduced to the case $C = 1$ by a change of the length of the unit in R_y. This completes the proof of Theorem XIX, since if $C = 0$, then $\phi = \chi$, and χ is a convex distribution function.

5. *Cauchy's transcendents and their generalizations.* Theorems VIII and IX have not been proven in § 2. Theorems XX and XXI, to be proven in what follows, generalize Theorems VIII and IX to the case of an arbitrary dimension number.

THEOREM XX. *There exists for every dimension number* n *and for every positive* $\lambda \leqq 2$ *a convex distribution function* $\phi = \phi_\lambda = \phi_\lambda(E)$ *such that*

$$\Lambda(u;\phi_\lambda) = \exp(-|\,u\,|^\lambda). \tag{61}$$

Remark. If $\lambda > 2$, the distribution function ϕ_λ does not exist in virtue of Theorem XIV.

Proof. If $\lambda = 2$, the existence and the convexity of ϕ_λ is obvious; cf. the end of the Proof of Theorem XVIII. Let therefore $0 < \lambda < 2$. Since $\lambda > 0$, the integral of the positive function $\mathrm{Min}(1, |\,y\,|^{-n-\lambda})$ over the whole space R_y

is finite, and so there exists a distribution function $\psi_\lambda = \psi_\lambda(E)$ the density of which is

(62) $$A_{n\lambda}\operatorname{Min}(1, |y|^{-n-\lambda}), \text{ where } A_{n\lambda} > 0,$$

the factor $A_{n\lambda}$ of proportionality being determined by the condition $\psi_\lambda(R_y) = 1$. Let ϑ denote the angle between the vectors u and y, so that

$$uy = sr\cos\vartheta, \text{ where } s = |u|,\ r = |y|,\ 0 \leqq \vartheta \leqq \pi.$$

Since

$$\Lambda(u;\psi_\lambda) = A_{n\lambda}\{\int_{|y|\leqq 1} e^{iuy}\mu(dR_y) + \int_{1\leqq|y|} e^{iuy}|y|^{-n-\lambda}\mu(dR_y)\}$$

in view of (47), (49) and (62), on introducing into R_y polar coördinates and denoting by B_n a positive constant which depends only on the dimension number n, it is seen that

$$\Lambda(u;\psi_\lambda) = A_{n\lambda}\Big\{B_n\int_0^1\Big[\int_0^\pi \exp(i|u|r\cos\vartheta)\sin^{n-2}\vartheta d\vartheta\Big] nr^{n-1}dr$$
$$+ B_n\int_1^{+\infty}\Big[\int_0^\pi \exp(i|u|r\cos\vartheta)\sin^{n-2}\vartheta d\vartheta\Big] nr^{n-1}r^{-n-\lambda}dr\Big\}.$$

It will be convenient to use the abbreviation

(63) $$\{J_\nu(z)\} = 2^\nu\Gamma(\nu+1)z^{-\nu}J_\nu(z), \qquad \nu \geqq -\tfrac{1}{2},$$

so that $\{J_\nu(x)\}$ is an even entire function. Since both inner integrals represent $C_n\{J_{-1+n/2}(|u|r)\}$, where C_n is a positive constant, it follows by placing $\alpha_{n\lambda} = A_{n\lambda}B_n nC_n$ that

(64) $$\Lambda(u;\psi_\lambda)/\alpha_{n\lambda} = \int_0^1 r^{n-1}\{J_{-1+n/2}(|u|r)\}dr + \int_1^{+\infty}\{J_{-1+n/2}(|u|r)\}r^{-1-\lambda}dr,$$

where $\alpha_{n\lambda}$ is a positive constant. So far it has been tacitly assumed that $n > 1$, since if $n = 1$, there is no polar angle ϑ varying from $\vartheta = 0$ to $\vartheta = \pi$. It is, however, clear from (38) and (63) that (64) holds for $n = 1$ also. Since by (63) and by the integral definition of the Bessel function

(65) $$\{J_\nu(z)\} = \pi^{-1/2}(\Gamma(\nu+1)/\Gamma(\nu+\tfrac{1}{2}))\int_{-1}^1 e^{iz\theta}(1-\theta^2)^{\nu-1/2}d\theta,$$

if $\nu > -\frac{1}{2}$, it is clear that

(65a) $$|\{J_\nu(x)\}| \leqq \{J_\nu(0)\} > 0, \quad -\infty < x < +\infty,$$

and (65a) holds, in view of (38), for $\nu = -\frac{1}{2}$ also.

Put

(66) $$f_{\lambda n}(q) = \int_q^{+\infty} v^{-1-\lambda}[\{J_{-1+n/2}(0)\} - \{J_{-1+n/2}(v)\}]dv, \text{ where } q > 0.$$

Since $\{J_\nu(z)\}$ is an even entire function, the integrand in (66) is, as $v \to +0$,

$$v^{-1-\lambda}O(v^2) = O(v^{1-\lambda}) = O(v^{-1+\epsilon}),$$

where $\epsilon > 0$ in virtue of $\lambda < 2$. Thus there exists a finite limit $f_{\lambda n}(+0)$. Furthermore, this limit is positive. In fact, the integrand in (66) is almost everywhere positive in view of (65a), which means that $f_{\lambda n}(q)$ is a steadily decreasing positive function of $q > 0$, so that the limit $f_{\lambda n}(+0)$ also is positive. Thus, on placing $\beta_{n\lambda} = f_{\lambda n}(+0)$,

(67) $$|u|^\lambda f_{\lambda n}(|u|) = \beta_{n\lambda}|u|^\lambda + o(|u|^\lambda) \text{ as } |u| \to 0,$$

where $\beta_{n\lambda} > 0$. Similarly, if

(68) $$h_n(q) = q^{-n}\int_0^q v^{n-1}[\{J_{-1+n/2}(0)\} - \{J_{-1+n/2}(v)\}]dv, \text{ where } q > 0,$$

then, as $q \to +0$,

$$h_n(q) = q^{-n}\int_0^q v^{n-1}\,O(v^2)dv = O(q^2),$$

and so, since $\lambda < 2$,

(69) $$h_n(|u|) = o(|u|^\lambda) \text{ as } |u| \to 0.$$

On choosing in (64) the point u as the origin of R_u and combining the resulting relation with (64) itself, it is seen that

(70) $$1/\alpha_{n\lambda} - \Lambda(u;\psi_\lambda)/\alpha_{n\lambda}$$

is the sum of

(71) $$\int_0^1 r^{n-1}\{J_{-1+n/2}(0)\}dr - \int_0^1 r^{n-1}\{J_{-1+n/2}(|u|r)\}dr$$

and

(72) $$\int_1^{+\infty}\{J_{-1+n/2}(0)\}r^{-1-\lambda}dr - \int_1^{+\infty}\{J_{-1+n/2}(|u|r)\}r^{-1-\lambda}dr.$$

On placing $v = |u|r$, where $|u| \neq 0$ is fixed, (71) and (72) appear in the respective forms

$$|u|^{-n}\int_0^{|u|} v^{n-1}[\{J_{-1+n/2}(0)\} - \{J_{-1+n/2}(v)\}]dv = h_n(|u|)$$

and

$$|u|^\lambda\int_{|u|}^{+\infty}[\{J_{-1+n/2}(0)\} - \{J_{-1+n/2}(v)\}]v^{-1-\lambda}dv = |u|^\lambda f_{\lambda n}(|u|)$$

in view of (68) and (66). Hence the difference (70) is

$$= h_n(|u|) + |u|^\lambda f_{\lambda n}(|u|),$$

and so, according to (69) and (67),

$$= o(|u|^\lambda) + \beta_{n\lambda} |u|^\lambda + o(|u|^\lambda) \text{ as } |u| \to 0.$$

Consequently, on denoting by $\gamma = \gamma_{n\lambda}$ the product of the positive constants $\alpha_{n\lambda}$ and $\beta_{n\lambda}$ and placing $a = \gamma^{1/\lambda} > 0$,

$$(73) \qquad \Lambda(u; \psi_\lambda) = 1 - a^\lambda |u|^\lambda + o(|u|^\lambda) \text{ as } |u| \to 0; \ a > 0.$$

It follows that ψ_λ may be applied, in the manner of Lie, as an infinitesimal generator of the distribution function ϕ_λ the existence and convexity of which are to be proven. For let $\psi_{m\lambda}$ denote the distribution function defined by

$$(74) \qquad \psi_{m\lambda}(E) = \psi_\lambda(am^{1/\lambda}E) * \psi_\lambda(am^{1/\lambda}E) * \cdots * \psi_\lambda(am^{1/\lambda}E),$$

where the "factor" ψ_λ occurs m times. Then from (48)

$$\Lambda(u; \psi_{m\lambda}) = [\Lambda(a^{-1}m^{-1/\lambda}u; \psi_\lambda)]^m,$$

since the Fourier transform of $\phi(cE)$ is $\Lambda(u/c; \phi)$. Consequently, from (73),

$$\Lambda(u; \psi_{m\lambda}) = [1 - m^{-1} |u|^\lambda + o(m^{-1} |u|^\lambda)]^m, \ |u| \to 0.$$

Hence it is clear from a standard property of the exponential function that

$$\Lambda(u; \psi_{m\lambda}) \to \exp(-|u|^\lambda), \ m \to +\infty,$$

holds uniformly in every fixed sphere $|u| <$ Const. of R_u. It therefore follows from the continuity theorem mentioned at the beginning of § 4 that there exists a distribution function $\phi_\lambda = \lim \psi_{m\lambda}$ for which $\Lambda(u; \phi_\lambda)$ is the limit of $\Lambda(u; \psi_{m\lambda})$ as $m \to +\infty$. Thus ϕ_λ satisfies (61). Furthermore, $\phi_\lambda = \lim \psi_{m\lambda}$ is, according to Theorem XII, certainly convex if every $\psi_{m\lambda}$ is convex, while (74) is, according to Theorem XIII, certainly convex if $\psi_\lambda(cE)$, where $c = a^{-1}m^{1/\lambda}$, is convex, i. e., if $\psi_\lambda(E)$ is convex. Now $\psi_\lambda(E)$ has been defined as the absolutely continuous distribution function the density of which is (62), and (62) clearly is a non-increasing function of $|y|$; hence $\psi_\lambda(E)$ is convex (cf. § 4). This completes the proof of Theorem XX.

Since the integral of (61) over the whole space R_u is finite, (51) is applicable. Hence the density of $\phi_\lambda(E)$ is $A_n F_{n\lambda}(|y|)$, where $A_n > 0$ and

$$(75) \qquad F_{n\lambda}(x) = x^{1-n/2} \int_0^{+\infty} s^{n/2} \exp(-s^\lambda) J_{-1+n/2}(xs)\,ds = F_{n\lambda}(-x),$$

since $\{J_\nu(z)\}$ is an even function. If $n = 1$, this density goes over into (28) in virtue of (38). Hence Theorem IX is implied by

THEOREM XXI. *The even function* (75), *where* $n = 1, 2, \cdots$, *has for* $-\infty < x < +\infty$ *derivatives of arbitrarily high order and is regular at every real* $x \neq 0$. *If* $0 < \lambda < 1$, *there is at* $x = 0$ *a singularity which has the*

character of the one described in Theorem IX. *If* $\lambda = 1$, *the function* $F_{n\lambda}(z) = F_{n\lambda}(x + iy)$ *is regular in a strip* $|y| <$ const. *but not in the whole plane, while if* $\lambda > 1$, *it is a transcendental entire function of order* $(1 - \lambda^{-1})^{-1}$. *Finally, the function has no real zero at all or has real zeros of odd multiplicity according as* $0 < \lambda \leqq 2$ *or* $\lambda > 2$.

Remark. The proof of the last statement is, in view of Theorems XX and XIV, exactly the same as that given in the case $n = 1$ in § 2. Except for the last statement of Theorem XXI, it will be needless to assume that n is an integer. On placing $\nu = -1 + \frac{1}{2}n$ and using the abbreviation (63), the function (75) is

$$(76) \qquad G_{\nu\lambda}(z) = \int_0^{+\infty} \{J_\nu(zs)\} s^{2\nu+1} \exp(-s^\lambda)\, ds$$

up to a constant positive factor. In what follows, instead of $\nu = -\frac{1}{2}, 0, \frac{1}{2}, 1, \cdots$, merely $\nu \geqq -\frac{1}{2}$ will be asssumed, while, as before, $\lambda > 0$.

Proof. It is clear from (65a) that the integral

$$(77) \qquad H_{\nu\lambda}(z) = \int_0^{+\infty} \{J_\nu(s)\} s^{2\nu+1} \exp(-s^\lambda/z^\lambda)\, ds$$

is, for every fixed real or complex $z \neq 0$, majorized by the integral

$$(78) \qquad \{J_\nu(0)\} \int_0^{+\infty} s^{2\nu+1} \,|\exp(-s^\lambda/z^\lambda)|\, ds.$$

Let W_λ denote the infinite open wedge

$$(79) \qquad W_\lambda: \quad -\tfrac{1}{2}\pi/\lambda < \operatorname{arc} z < \tfrac{1}{2}\pi/\lambda, \text{ where } z \neq 0.$$

Thus W_λ is simply connected but not necessarily schlicht; in fact, the number of its sheets becomes infinite as $\lambda \to 0$. If $\lambda < 1$, the half-plane consisting of all numbers $z \neq 0$ which have a non-negative real part lies within W_λ. Let z^λ be defined in W_λ as the univalued and continuous function of the position which is positive along the half-line arc $z = 0$. Then the integral (78) is uniformly convergent in every fixed closed and bounded sub-set of (79). Hence, since (77) is majorized by (78) and z^λ is regular on W_λ, the integral (77) represents a regular function $H_{\nu\lambda}(z)$ on W_λ. As far as the integral (76) is concerned, one cannot say more than that it is uniformly convergent along the real axis, a property which does not indicate the analyticity of the function $G_{\nu\lambda}(z)$. If $0 < \lambda < 1$, it is clear from

$$\log \operatorname*{Max}_{|z| \leqq r} |\{J_\nu(z)\}| = \log \{J_\nu(ir)\} \sim r; \quad r \to \infty,$$

that the integral (76) is divergent at all points $z \neq 0$ of the imaginary axis $\operatorname{arc} z = \pm \frac{1}{2}\pi$, points which are contained in W_λ if $\lambda < 1$. However,

$$G_{\nu\lambda}(z) = z^{-2\nu-2}H_{\nu\lambda}(z) \tag{80}$$

for every $z > 0$, as seen from (76) and (77) by a change of the integration variable. Since $H_{\nu\lambda}(z)$ is a regular function on W_λ, the relation (80) gives a regular analytic continuation of $G_{\nu\lambda}(z)$, where $0 < z < +\infty$, into W_λ. This holds for every $\lambda > 0$. Since $\{J_\nu(z)\} = \{J_\nu(-z)\}$, the function (76) also is even, and so it is not necessary to consider its analytic continuation which belongs to $-\infty < z < 0$.

It is clear from (65) and (38) that the m-th derivative of $\{J_\nu(x)\}$ is bounded for $-\infty < x < +\infty$, where m is arbitrarily fixed. Hence the integral resulting from (76) by differentiating the integrand m times with respect to z is uniformly convergent for all real z. Thus if z is real, all derivatives of the function (76) exist and may be obtained by formal differentiation; in particular,

$$G_{\nu\lambda}^{(m)}(0) = \{J_\nu(0)\}^{(m)} \int_0^{+\infty} s^m s^{2\nu+1} \exp(-s^\lambda)\, ds \tag{81}$$

for every m, it being understood that these derivatives are obtained by restricting z to the real axis. On combining this with the analytic continuation obtained before, it follows that Theorem XXI will be proven if one shows that the radius of convergence of the power series

$$\sum_{m=0}^{\infty} m!^{-1} G_{\nu\lambda}^{(m)}(0) z^m \tag{82}$$

is zero, finite and positive or infinite according as $0 < \lambda < 1$, $\lambda = 1$ or $\lambda > 1$, and that the order is $(1 - \lambda^{-1})^{-1}$ in the latter case. In fact, if $\lambda \geqq 1$, then the integral (76) clearly is uniformly convergent at least in a small circle $|z| <$ const., so that the series (82) necessarily represents the function $G_{\nu\lambda}(z)$ within the circle of convergence.

Now if $\nu > -\frac{1}{2}$, then from (65)

$$\{J_\nu(0)\}^{(m)} = \pi^{-1/2}(\Gamma(\nu+1)/\Gamma(\nu+\tfrac{1}{2}))i^m \int_{-1}^{1} \theta^m(1-\theta^2)^{\nu-1/2} d\theta;$$

hence

$$\{J_\nu(0)\}^{(2m+1)} = 0, \tag{83}$$

while

$$\{J_\nu(0)\}^{(2m)} = (-1)^m \pi^{-1/2}\Gamma(\nu+1)\Gamma(m+\tfrac{1}{2})/\Gamma(m+\nu+1),$$

as seen by introducing into $\int_{-1}^{1} = 2\int_0^1$ the integration variable θ^2. It is

clear from (38) and (63) that (83) holds for $\nu = -\frac{1}{2}$ also. On the other hand, the factor of $\{J_\nu(0)\}^{(m)}$ in (81) is

$$\int_0^{+\infty} s^{(m+2\nu+1)/\lambda} \exp(-s)\lambda^{-1}s^{-1+1/\lambda}\, ds = \lambda^{-1}\Gamma(2\lambda^{-1}(\tfrac{1}{2}m + \nu + 1)).$$

Hence

$$(84) \quad G_{\nu\lambda}{}^{(2m)}(0) = (-1)^m g_{\nu\lambda}\Gamma(2\lambda^{-1}(m+\nu+1))\Gamma(m+\tfrac{1}{2})/\Gamma(m+\nu+1),$$

where

$$g_{\nu\lambda} = \pi^{-1/2}\lambda^{-1}\Gamma(\nu+1)$$

is positive and independent of m; furthermore, $G_{\nu\lambda}{}^{(2m+1)}(0) = 0$. Now it is clear from (84) and from Stirling's formula that

$$|\, G_{\nu\lambda}{}^{(2m)}(0)\,|^{1/(2m)} \sim [\Gamma(2m/\lambda)]^{1/(2m)} \quad \text{as } m \to \infty.$$

Consequently, the reciprocal value of the radius of convergence of (82) is

$$\limsup_{m=\infty} |\, G_{\nu\lambda}{}^{(m)}(0)/m!\,|^{1/m} = \lim_{m=\infty} [\Gamma(2m/\lambda)/\Gamma(2m)]^{1/(2m)}$$

and this is, by Stirling's formula, 0, 1, or $+\infty$ according as $\lambda > 1$, $\lambda = 1$ or $\lambda < 1$. Finally, it is known that the order of an entire function $f(z)$ may be obtained merely from the sequence of its Taylor coefficients by calculating

$$\limsup_{m=\infty} (m \log m)/\log |\, m!/f^{(m)}(0)\,|.$$

This completes the proof of Theorem XXI, since $G_{\nu\lambda}{}^{(2m+1)}(0) = 0$, while

$$\log |(2m)!/G_{\nu\lambda}{}^{(2m)}(0)| \sim \log [\Gamma(2m)/\Gamma(2m/\lambda)] \sim (1-\lambda^{-1})[2m \log(2m)],$$

as seen from (84) and from Stirling's formula.

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ON THE ASYMPTOTIC DISTRIBUTION OF ALMOST PERIODIC FUNCTIONS WITH LINEARLY INDEPENDENT FREQUENCIES.

By RICHARD KERSHNER and AUREL WINTNER.

Let

$$x(t) \sim \sum_{m=0}^{\infty} a_m \cos \lambda_m (t - \delta_m), \text{ where } a_m \geqq 0 \text{ and } 0 \leqq \lambda_m \delta_m < 2\pi,$$

be almost periodic in the sense of Bohr and let $A \leqq x \leqq B$ denote the least closed interval containing all values x attained by $x(t)$ for $-\infty < t < +\infty$. It has been proven [1] that $x(t)$ always posseses an asymptotic distribution function $\sigma(x)$, $-\infty < x < +\infty$, and that this monotone function $\sigma(x)$ is nowhere constant in the interval $A \leqq x \leqq B$, while, of course, $\sigma(x) \equiv 0$ if $x < A$ and $\sigma(x) \equiv 1$ if $x > B$. The present note deals with the important particular case where $a_m \neq 0$ for every m and the frequencies $\lambda_0, \lambda_1, \cdots$ are linearly independent, so that

$$-A = B = \sum_{m=0}^{\infty} a_m < +\infty \qquad (a_m > 0)$$

by a well-known theorem of Bohr.

On denoting by $\rho(x)$, $-\infty < x < +\infty$, the monotone continuous function which is 0, 1 or $1 - \pi^{-1} \arccos x$ according as x is on the left, on the right or in the interior of the interval $-1 < x < 1$ and placing, for $-\infty < x < +\infty$,

$$\sigma_{n+1}(x) = \int_{-\infty}^{+\infty} \sigma_n(x - \xi)\, d\rho_{n+1}(\xi), \tag{1}$$

$$\text{where } \rho_n(x) = \rho(x/a_n) \text{ and } \sigma_0(x) = \rho_0(x),$$

the asymptotic distribution function $\sigma(x)$ of $x(t)$ may be obtained from [2]

$$\sigma(x) = \lim_{n=+\infty} \sigma_n(x) \tag{2}$$

and has, for $-\infty < x < +\infty$, derivatives of arbitrarily high order.[2] Hence there arises the question of whether $\sigma(x)$ is analytic.[3] There is a general method [4] by which the analyticity of a distribution function $\sigma(x)$ can be proven in some cases but this method breaks down in the present case. For the

[1] A. Wintner, *Spektraltheorie der unendlichen Matrizen*, Leipzig, 1929, pp. 254-255 and p. 269.

[2] A. Wintner, *American Journal of Mathematics*, vol. 55 (1933), pp. 312-316.

[3] Cf. *ibid.*, p. 310.

[4] A. Wintner, *American Journal of Mathematics*, vol. 56 (1934), p. 659.

method in question yields the analyticity of $\sigma(x)$ either for every x or for no x. Now the present $\sigma(x)$ cannot be analytic at the end points of the interval

$$(3) \qquad -\sum_{m=0}^{\infty} a_m \leqq x \leqq \sum_{m=0}^{\infty} a_m.$$

In fact, $\sigma(x)$ is nowhere constant in this interval but is constant both on the left and on the right of it, so that all derivatives of $\sigma(x)$ vanish at both end points of (3). The object of the present note is *to delimit in the range* (3) *of* $x(t)$ *intervals within which the asymptotic distribution function* $\sigma(x)$ *is regular analytic.*

First, $\sigma(x)$ *is regular in the interval*

$$(4) \qquad -a_0 + \sum_{m=1}^{\infty} a_m < x < a_0 - \sum_{m=1}^{\infty} a_m$$

whenever $a_0 - \sum_{m=1}^{\infty} a_m > 0$.

In order to prove this, put

$$(5) \qquad q_{n+1} = q_n + a_{n+1}, \text{ where } q_0 = 0,$$

so that

$$(6) \qquad 0 < q_n < q_{n+1} \rightarrow q, \text{ where } q = \sum_{m=1}^{\infty} a_m.$$

Thus $a_0 - q > 0$ by assumption. Let ϵ be an arbitrarily small fixed positive number less than $a_0 - q$. On removing from a complex z-plane the half-lines $-\infty < z \leqq -1$ and $1 \leqq z < +\infty$, there results a simply connected schlicht domain in which the function $F(z) = (1 - z^2)^{-\frac{1}{2}}$, where $F(0) = +1$, is univalued and regular. If z is in the circle $|z| < a_0 - \epsilon$, the function $\sigma_0(z)$ defined by

$$\sigma_0(z) = (\pi a_0)^{-1} \int_{-a_0}^{z} F(\zeta/a_0)\, d\zeta$$

is regular and $|\sigma_0(z)|$ has in this circle a finite least upper bound M, since $\epsilon > 0$ is fixed. It is clear from (6) and from the assumption $\epsilon < a_0 - q$ that $a_0 - q_n - \epsilon > 0$, i. e., that $|z| < a_0 - q_n - \epsilon$ is a circle for every n. Suppose that, for a given $n = \bar{n}$, there has been defined in the circle $|z| < a_0 - q_n - \epsilon$ a function $\sigma_n(z)$ in such a way that $\sigma_n(z)$ is regular and $|\sigma_n(z)| \leqq M$ in this circle. This condition has been satisfied for $n = 0$, since $q_0 = 0$ by (5). If $n = \bar{n}$ is arbitrarily given and if $|z| < a_0 - q_{n+1} - \epsilon$ and $-a_{n+1} < \xi < a_{n+1}$, then $|z - \xi| < a_n - q_n - \epsilon$ in view of (5). Hence one may define in the circle $|z| < a - q_{n+1} - \epsilon$ a function $\sigma_{n+1}(z)$ by placing

$$\sigma_{n+1}(z) = (\pi a_{n+1})^{-1} \int_{-a_{n+1}}^{a_{n+1}} \sigma_n(z - \xi) F(\xi/a_{n+1})\, d\xi.$$

Then, since $\sigma_n(z)$ is regular and $|\sigma_n(z)| \leqq M$ in the circle $|z| < a - q_n - \epsilon$, it is clear that $\sigma_{n+1}(z)$ is regular and

$$|\sigma_{n+1}(z)| \leqq (\pi a_{n+1})^{-1} \int_{-a_{n+1}}^{a_{n+1}} M \, | F(\xi/a_{n+1}) | \, d\xi = M\pi^{-1} \int_{-1}^{1} (1-\xi^2)^{-\frac{1}{2}} \, d\xi = M$$

in the circle $|z| < a_0 - q_{n+1} - \epsilon$.

Thus there has been defined a sequence $\{\sigma_n(z)\}$ such that $\sigma_n(z)$ is regular and $|\sigma_n(z)| \leqq M$ in the circle $|z| < a_0 - q_n - \epsilon$, where $n = 0, 1, \cdots$. Hence it is seen from (6) that the functions $\sigma_n(z)$ are regular and uniformly bounded in the circle $|z| < a_0 - q - \epsilon$ which is independent of n. Now it is clear from the successive definition of the complex functions $\sigma_n(z)$ that, when z is real, they are identical with the functions defined by (1), so that $\{\sigma_n(z)\}$ is convergent for real z in view of (2). It follows, therefore, from the elements of the theory of normal families (Vitali) that $\{\sigma_n(z)\}$ is uniformly convergent in every closed subset of the circle $|z| < a_0 - q - \epsilon$. Since ϵ is arbitrarily small, the limit function is regular in the circle $|z| < a_0 - q$. In particular, (2) is regular in the interval $-a_0 + q < x < a_0 - q$ or, by the definition (6) of q, in the interval (4); q. e. d.

The assumption of the result thus proven was that the amplitudes a_m of $x(t)$ satisfy the condition

$$\text{(7)} \qquad a_n > \sum_{m=n+1}^{\infty} a_m$$

for $n = 0$. If (7) holds not only for $n = 0$ but for $n = 1$ also, an iteration[5] of the above argument shows that $\sigma(x)$ is regular analytic not only in the interval (4) but also in each of the two additional subintervals

$$\pm a_0 - a_1 + \sum_{m=2}^{\infty} a_m < x < \pm a_0 + a_1 - \sum_{m=2}^{\infty} a_m$$

of (3) which have no point in common with each other or with (4). It suffices to notice that the singular points of $\sigma_1(x)$ are the four points $\pm a_0 \pm a_1$, the signs being this time independent of each other. Similarly, if (7) holds for $n = 0, 1, \cdots, N$, one obtains $2^{N+1} - 1$ disjoint open subintervals of (3) such that (2) is regular analytic within each of these $2^{N+1} - 1$ intervals.

In order to describe the situation in the case $N = +\infty$ where (7) holds for every n, let the interval (3) be decomposed into two complementary sets S, T by placing a point x of (3) into S or into T according as x can or cannot be represented in at least one way in the form

[5] Cf. R. Kershner and A. Wintner, *American Journal of Mathematics*, vol. 57 (1935), pp. 544-545.

$$x = \sum_{m=0}^{\infty} \pm a_m,$$

where the signs are independent of each other. Then S is [6] perfect and nowhere dense, and so T is open and everywhere dense in the interval (3). Now it is easily seen that the intervals within which the indefinite iteration of the above argument yields the analyticity of the function (2) are exactly the intervals which constitute the open set T. *Thus if the amplitudes a_m of $x(t)$ satisfy the condition (7) for every n, then the asymptotic distribution $\sigma(x)$ is regular analytic in infinitely many disjoint open intervals which lie dense in the range* (3) *of $x(t)$.* It remains undecided whether or not the complementary set S, consisting of the cluster points of the end points of these open intervals of regularity, actually contains but singular points of $\sigma(x)$. All that is certain is that all derivatives of $\sigma(x)$ exist at every point of S also and that the set of the 2^{n+1} singular points $x = \pm a_0 \pm \cdots \pm a_n$ of $\sigma_n(x)$ tends to the set S as $n \rightarrow +\infty$, finally that the clustering of these singular points is strongest at the two end points of the interval (3), at which points $\sigma(x)$ certainly is not regular analytic (cf. the introduction). It may be mentioned [7] that the measure of the nowhere dense perfect set S is zero or positive according as $2^n a_n$ does or does not tend to zero as $n \rightarrow \infty$. If, for instance, $a_n = a^n$, where $0 < a < \frac{1}{2}$, then (7) holds for every n and S is of measure zero. This example is of interest in view of the non-differentiable function of Weierstrass.[8] If $a_n = 2^{-n} + 3^{-n}$, then (7) is satisfied for every n and S has a positive measure.

So far it has been assumed that $x(t)$ is real-valued. The case

$$x(t) + iy(t) \sim \sum_{n=0}^{\infty} a_n \exp i\lambda_n(t - \delta_n), \quad -\infty < t < +\infty,$$

of a complex-valued function offers no new problem, since it may be reduced to the case $y(t) \equiv 0$ treated above by means of an integral equation of the Abel type,[9] the frequencies λ_n being linearly independent. It is expected that the unsolved problems of analyticity in the general case of linearly independent moduli [10] may also be treated along the lines of the present note, the complications involved being but technical in nature.

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[6] Cf., e. g., *loc. cit.* [5]

[7] Cf. *loc. cit.* [5]

[8] Cf. A. Wintner, *American Journal of Mathematics*, vol. 55 (1933), pp. 603-605.

a positive measure.

[9] Cf. *loc. cit.* [2], pp. 316-319.

[10] Cf. *loc. cit.* [2], pp. 328-329 and, for a complete theory, B. Jessen and A. Wintner, *Transactions of the American Mathematical Society*, vol. 38 (1935), pp. 48-88.

NECESSARY AND SUFFICIENT CONDITIONS FOR POTENTIALS OF SINGLE AND DOUBLE LAYERS.[1]

By George A. Garrett.

1. *Introduction.* We consider the problem of obtaining necessary and sufficient conditions that a harmonic function defined in a three-dimensional region T be such that it can be represented in the region as a potential function. Denoting the harmonic function by $v(M)$ or $U(M)$, we find conditions necessary and sufficient in order that the function be representable as the potential of a single-layer distribution,

$$v(M) = \int_{S_0} \frac{1}{MP}\, d\nu(e_P)$$

and as the potential of a double-layer distribution,

$$U(M) = \int_{S_0} \frac{\cos \sphericalangle\, (MP, n_P)}{\overline{MP}^2}\, d\nu(e_P)\,;$$

where S_0 is the boundary of T; $\nu(e)$ is the mass function, a completely additive function of point sets defined on S_0; P is a point of S_0; and n_P is the interior normal to S_0 at P.

These conditions are given in terms of " normal families " of surfaces.

2. *Potentials and normal families of surfaces.*

Definition. We shall call a family $\{S\}$ of simple closed surfaces having a tangent plane at every point a normal family provided that they lie interior or exterior to the surface S_0, which we shall consider as a member of the family, and

(a) for each surface of the family and for any two points A, B of the surface

$$|\sphericalangle\, (n_A, n_B)| < \gamma\, \overline{AB}$$

where γ is a constant, n_A is the normal at A, and n_B is the normal at B;

[1] This paper is an extension of the corresponding treatment for the two-dimensional case as presented by Professor G. C. Evans in a seminar conducted at The Rice Institute during the spring of 1933. Theorems 4. 1 and 4. 2 overlap somewhat a theorem published by Ch. J. De la Vallée Poussin in November of the same year. Cf. De la Vallée Poussin, " Propriétés des Fonctions Harmoniques dans un Domaine Ouvert Limité par des Surfaces à Courbure Bornée," *Annali della Rendiconti Scuola Normale Superiore de Pisa*, serie 2, vol. 2 (1933-XI), pp. 167-199. A more complete reference to this theorem of De la Vallée Poussin's is given in the footnote to Theorem 4. 2.

(b) for every two surfaces S_1 and S_2 of the family

$$|\sphericalangle (n_{A_1}, n_{A_2})| < \gamma\, \overline{A_1 A_2}$$

where n_{A_1} is the normal to S_1 at A_1 on S_1, and n_{A_2} is the normal to S_2 at A_2 on S_2;

(c) in any infinite subset of the family there is a subsequence $\{S'_i\}$ which approaches a surface S of the family, approach being in the sense that the maximum normal distance from S approaches zero.

We take the positive directions of the normals to the surfaces to be toward the interiors of the surfaces. We denote by T^+ and T^- the regions interior and exterior respectively to S_0.

Consider a surface S_0 having the property (a). Let P be a point of S_0 and M a point not on S_0 and such that the normal to S_0 at the point Q on S_0 passes through M. Let M' be the image of M in the tangent plane to S_0 at Q. Consider a normal family $\{S\}$ of surfaces inside or outside S_0 and let τ be the maximum normal distance of a member of the family from S_0. We suppose that τ is small enough so that there is a unique $1:1$ correspondence [2] of points A_0 of S_0 along the normals n_{A_0} with points A of an arbitrary surface of $\{S\}$. If M is on S, then we denote by n_M the normal to S at M. We suppose also that τ is so small that dS_Q/dS_M is bounded away from zero and infinity, where dS_Q and dS_M are the elements of area of S_0 at Q and S at M, respectively.

We write

$$\begin{aligned} \phi &= \sphericalangle (MP, n_P); & \phi' &= \sphericalangle (M'P, n_P) \\ \theta_M &= \sphericalangle (MP, n_M); & \theta'_M &= \sphericalangle (M'P, n_{M'}) \\ \theta_Q &= \sphericalangle (MP, n_Q); & \theta'_Q &= \sphericalangle (M'P, n_Q) \\ r &= MP; & r' &= M'P. \end{aligned}$$

By the symbol θ without subscripts we shall mean both θ_M and θ_Q; and by θ', both θ'_M and θ'_Q. We consider the functions

$$\text{(I).} \qquad U(M) = \int_{S_0} \frac{\cos\phi}{r^2}\, d\nu(e_P),$$

$$\text{(II).} \qquad U_1(M) = \int_{S_0} \frac{\cos\theta}{r^2}\, d\nu(e_P),$$

$$\text{(III).} \qquad U_2(M) = \int_{S_0} \frac{\sin\phi}{r^2}\, d\nu(e_P),$$

$$\text{(IV).} \qquad U_3(M) = \int_{S_0} \frac{\sin\theta}{r^2}\, d\nu(e_P).$$

[2] That this is true for τ small enough is shown by G. C. Evans and E. R. C. Miles in "Potentials of general masses in single and double layers. The relative boundary value problems," *American Journal of Mathematics*, vol. 53 (1931), pp. 493-516. Cf. p. 497. Hereafter we refer to this article as (A).

Let n_Q be the z-axis and the tangent plane to S_0 at Q be the xy-plane for a system of rectangular coördinates. Let s be the curve of intersection of S_0 and the plane determined by P and n_Q. We shall also denote by s the length of the arc QP measured along this curve. Take as the x-axis the tangent to s at Q. Let P have cylindrical coördinates (ρ, μ, z). Let A be any point of S_0. Then we denote by $S_0(\delta, A)$ the portion of S_0 containing A which is contained in the sphere of center A and radius δ; and by $S(\delta, A)$ the portion of S nearest to $S_0(\delta, A)$ which is cut out by this sphere. We suppose that P is in $S_0(\delta, Q)$ and that δ is so small that

$$s < 2\rho, \qquad \phi, \phi' \leq \pi.$$

In the inequalities which follow we may suppose without loss of generality that $r \leq r'$. For P in $S_0(\delta, Q)$ we have

$$
(2.1)\quad \left\{
\begin{aligned}
& QM = QM' \leq 2r; \\
& |\phi - \theta_Q| \leq \sphericalangle(n_P, n_Q); \quad |\phi' - \theta'_Q| \leq \sphericalangle(n_P, n_Q); \\
& |\theta_M - \theta_Q| \leq \sphericalangle(n_M, n_Q); \quad |\theta'_M - \theta'_Q| \leq \sphericalangle(n_{M'}, n_Q); \\
& |\cos\phi - \cos\theta_Q| \leq 2\left|\sin\frac{\phi - \theta_Q}{2}\right| < 2\gamma\rho; \\
& |\cos\phi' - \cos\theta'_Q| \leq 2\left|\sin\frac{\phi' - \theta'_Q}{2}\right| < 2\gamma\rho; \\
& |\sin\phi - \sin\theta_Q| \leq 2\left|\sin\frac{\phi - \theta_Q}{2}\right| < 2\gamma\rho; \\
& |\sin\phi' - \sin\theta'_Q| \leq 2\left|\sin\frac{\phi' - \theta'_Q}{2}\right| < 2\gamma\rho; \\
& |\sin\theta_Q - \sin\theta'_Q| \leq 2\left|\cos\frac{\theta_Q + \theta'_Q}{2}\right| \leq 2\,|\sin\sphericalangle(\rho, QP)| \\
& \qquad\qquad < \frac{2\gamma}{QP}\int_0^s s\,ds < 4\gamma\rho; \\
& |\cos\theta_Q + \cos\theta'_Q| \leq 2\left|\cos\frac{\theta_Q + \theta'_Q}{2}\right| < 4\gamma\rho; \\
& \left|\frac{1}{r^2} - \frac{1}{(r')^2}\right| = \left|\frac{\sin\theta_Q - \sin\theta'_Q}{\rho}\right| \left|\frac{1}{r} + \frac{1}{r'}\right| < 4\gamma\left(\frac{2}{r}\right) = \frac{8\gamma}{r}; \\
& |\sin\theta_M - \sin\theta_Q| \leq 2\left|\sin\frac{\theta_M - \theta_Q}{2}\right| < \gamma\,\overline{QM} \leq 2\gamma r; \\
& |\cos\theta_M - \cos\theta_Q| \leq 2\left|\sin\frac{\theta_M - \theta_Q}{2}\right| < 2\gamma r; \\
& |\sin\theta'_M - \sin\theta'_Q| \leq 2\left|\sin\frac{\theta'_M - \theta'_Q}{2}\right| < \gamma\,\overline{QM'} \leq 2\gamma r; \\
& |\cos\theta'_M - \cos\theta'_Q| \leq 2\left|\sin\frac{\theta'_M - \theta'_Q}{2}\right| < 2\gamma r.
\end{aligned}
\right.
$$

Since P is in $S_0(\delta, Q)$ if Q is in $S_0(\delta, P)$ the relations (2.1) subsist for Q in $S_0(\delta, P)$.

Now we wish to consider P as fixed and Q as a variable point in $S_0(\delta, P)$. Let n_P be the z'-axis and the tangent plane to S_0 at P be the $x'y'$-plane. Take the x'-axis along the tangent to the curve s^* at P, where s^* is the curve of intersection of S_0 and the plane determined by n_P and Q. We also denote by s^* the length of the arc QP measured along this curve. Let Q have cylindrical coördinates (ρ', μ', z'). Let $C(\delta, P)$ be the circle $\rho' = \delta$ in the $x'y'$-plane. Let δ be small enough so that the relations (2.1) hold and also small enough so that

$$s^* < 2\rho' \qquad | ds^* | < 2 \, | d\rho' |.$$

Hence we have

$$| dS_Q | < 2\rho' d\rho' d\mu' \qquad \tfrac{1}{2} < \rho/\rho' < 2.$$

LEMMA 1. *The integrals*

$$\int\limits_{S_0(\delta,P)} \left| \frac{\cos\theta}{r^2} + \frac{\cos\theta'}{(r')^2} \right| dS_Q; \quad \int\limits_{S_0(\delta,P)} \left| \frac{\sin\theta}{r^2} - \frac{\sin\theta'}{(r')^2} \right| dS_Q; \quad \int\limits_{S_0(\delta,P)} \left| \frac{\sin\phi}{r^2} - \frac{\sin\phi'}{(r')^2} \right| dS_Q;$$

$$\int\limits_{S_0(\delta,P)} \left| \frac{\cos\phi - \cos\theta}{r^2} \right| dS_Q; \quad \int\limits_{S_0(\delta,P)} \left| \frac{\cos\phi' - \cos\theta'}{(r')^2} \right| dS_Q; \quad \int\limits_{S_0(\delta,P)} \left| \frac{\cos\theta_M - \cos\theta_Q}{r^2} \right| dS_Q;$$

$$\int\limits_{S_0(\delta,P)} \left| \frac{\cos\theta'_M - \cos\theta'_Q}{(r')^2} \right| dS_Q; \quad \int\limits_{S_0(\delta,P)} \left| \frac{\sin\theta_M - \sin\theta_Q}{r^2} \right| dS_Q; \quad \int\limits_{S_0(\delta,P)} \left| \frac{\sin\theta'_M - \sin\theta'_Q}{(r')^2} \right| dS_Q;$$

all approach zero with δ, independently of τ. This statement is true if dS_Q and $S_0(\delta, P)$ are replaced by dS_M and $S(\delta, P)$ respectively.

Since dS_Q/dS_M is bounded away from zero and infinity, the second statement is a consequence of the first. Using θ_Q and θ'_Q for θ and θ' we have for the first integral

$$\begin{aligned}
\int\limits_{S_0(\delta,P)} \left| \frac{\cos\theta_Q}{r^2} + \frac{\cos\theta'_Q}{(r')^2} \right| dS_Q &= \int\limits_{S_0(\delta,P)} \left| \frac{\cos\theta_Q + \cos\theta'_Q}{r^2} - \cos\theta'_Q \left(\frac{1}{r^2} - \frac{1}{(r')^2} \right) \right| dS_Q \\
&\leq \int\limits_{S_0(\delta,P)} \left\{ \left| \frac{\cos\theta_Q + \cos\theta'_Q}{r^2} \right| + \left| \frac{1}{r^2} - \frac{1}{(r')^2} \right| \right\} dS_Q \\
&< 12\gamma \int\limits_{S_0(\delta,P)} \frac{dS_Q}{\rho} \leq 48\gamma \int\limits_{C(\delta,P)} d\rho' d\mu' = 96\pi\gamma\delta.
\end{aligned}$$

This inequality proves the lemma for the first integral. The four succeeding integrals are treated similarly, using θ_Q and θ'_Q for θ and θ'. For the sixth integral we have

$$\int\limits_{S_0(\delta,P)} \left| \frac{\cos\theta_M - \cos\theta_Q}{r^2} \right| dS_Q < 2\gamma \int\limits_{S_0(\delta,P)} \frac{dS_Q}{\rho} \leq 16\pi\gamma\delta.$$

The three remaining integrals are treated similarly. For the first integral using θ_M and θ'_M for θ and θ', we write

$$\int\limits_{S_0(\delta,P)} \left| \frac{\cos\theta_M}{r^2} + \frac{\cos\theta'_M}{(r')^2} \right| dS_Q \leq \int\limits_{S_0(\delta,P)} \left| \frac{\cos\theta_M - \cos\theta_Q}{r^2} \right| dS_Q$$
$$+ \int\limits_{S_0(\delta,P)} \left| \frac{\cos\theta'_M - \cos\theta'_Q}{(r')^2} \right| dS_Q + \int\limits_{S_0(\delta,P)} \left| \frac{\cos\theta_Q}{r^2} + \frac{\cos\theta'_Q}{(r')^2} \right| dS_Q.$$

The lemma has been proved for these last three integrals. We treat similarly the four succeeding integrals, using θ_M and θ'_M for θ and θ'.

As a corollary to Lemma 1, we have

LEMMA 2. *The integrals*

$$\int\limits_{S_0} \left| \frac{\cos\theta}{r^2} + \frac{\cos\theta'}{(r')^2} \right| dS_Q; \quad \int\limits_{S_0} \left| \frac{\sin\theta}{r^2} - \frac{\sin\theta'}{(r')^2} \right| dS_Q; \quad \int\limits_{S_0} \left| \frac{\sin\phi}{r^2} - \frac{\sin\phi'}{(r')^2} \right| dS_Q;$$
$$\int\limits_{S_0} \left| \frac{\cos\phi - \cos\theta}{r^2} \right| dS_Q; \quad \int\limits_{S_0} \left| \frac{\cos\phi' - \cos\theta'}{(r')^2} \right| dS_Q; \quad \int\limits_{S_0} \left| \frac{\cos\theta_M - \cos\theta_Q}{r^2} \right| dS_Q;$$
$$\int\limits_{S_0} \left| \frac{\cos\theta'_M - \cos\phi'_Q}{(r')^2} \right| dS_Q; \quad \int\limits_{S_0} \left| \frac{\sin\theta_M - \sin\theta_Q}{r^2} \right| dS_Q; \quad \int\limits_{S_0} \left| \frac{\sin\theta'_M - \sin\theta'_Q}{(r')^2} \right| dS_Q;$$

are bounded independently of τ. This statement is true if dS_Q and S_0 are replaced by dS_M and S respectively.

We denote by E_S a finite number of non-overlapping regions on S and let the corresponding regions on S_0 be E_{S_0}. We denote the measures of these sets by $m(E_S)$ and $m(E_{S_0})$.

THEOREM 2.1. *The absolute continuity of the integrals*

$$\int_{E_{S_0}} | U_3(M) - U_3(M') |\, dS_Q; \qquad \int_{E_{S_0}} | U_2(M) - U_2(M') |\, dS_Q;$$
$$\int_{E_S} | U_3(M) - U_3(M') |\, dS_M; \qquad \int_{E_S} | U_2(M) - U_2(M') |\, dS_M;$$

is uniform independently of τ.

It suffices to prove the theorem for the first two integrals, since dS_Q/dS_M is bounded away from zero and infinity. We write

$$I = \int_{E_{S_0}} |U_3(M) - U_3(M')|\, dS_Q \leq \int_{S_0} |d\nu(e_P)| \int_{E_{S_0}} \left| \frac{\sin\theta}{r^2} - \frac{\sin\theta'}{(r')^2} \right| dS_Q$$

$$\leq \int_{S_0} |d\nu(e_P)| \left\{ \int_{S_0(\delta,P)} \left| \frac{\sin\theta}{r^2} - \frac{\sin\theta'}{(r')^2} \right| dS_Q + \int_{E_{S_0} - E_{S_0}\cdot S_0(\delta,P)} \left| \frac{\sin\theta}{r^2} - \frac{\sin\theta'}{(r')^2} \right| dS_Q \right\}$$

We denote by $\bar{\nu}(S_0)$ the total variation of $\nu(e)$ over S_0. Since $\nu(e)$ is completely additive, $\bar{\nu}(S_0)$ is bounded. By Lemma 1, given $\epsilon > 0$, we may choose δ so small that the first of the inner integrals is less than $\epsilon/2\bar{\nu}(S_0)$. A positive constant K exists such that for Q in $E_{S_0} - E_{S_0} \cdot S_0(\delta, P)$

$$\left| \frac{\sin\theta}{r^2} - \frac{\sin\theta'}{(r')^2} \right| < K.$$

We now have

$$I < \epsilon/2 + Km(E_{S_0})\bar{\nu}(S_0).$$

Hence, for all E_{S_0} such that $m(E_{S_0}) \leq \epsilon/2K\bar{\nu}(S_0)$, I is less than ϵ. This is what was to be shown. The second integral may be treated similarly.

THEOREM 2.2. *The integrals*

$$\int_{S_0} |U_3(M) - U_3(M')|\, dS_Q; \qquad \int_{S_0} |U_2(M) - U_2(M')|\, dS_Q;$$
$$\int_{S} |U_3(M) - U_3(M')|\, dS_M; \qquad \int_{S} |U_2(M) - U_2(M')|\, dS_M$$

remain bounded as τ approaches zero and approach zero uniformly with τ.

Again it suffices to prove the theorem for the first two integrals and again the same method of proof is used for both. That the integrals remain bounded as τ approaches zero follows immediately from the preceding theorem. For the first integral we have

$$\int_{S_0} |U_3(M) - U_3(M')|\, dS_Q$$
$$\leq \int_{S_0} |d\nu(e_P)| \left\{ \int_{S_0 - S_0(\delta,P)} \left| \frac{\sin\theta}{r^2} - \frac{\sin\theta'}{(r')^2} \right| dS_Q + \int_{S_0(\delta,P)} \left| \frac{\sin\theta}{r^2} - \frac{\sin\theta'}{(r')^2} \right| dS_Q \right\}.$$

By Lemma 1 we may fix δ so small that, given any $\epsilon > 0$,

$$\int_{S_0(\delta,P)} \left| \frac{\sin\theta}{r^2} - \frac{\sin\theta'}{(r')^2} \right| dS_Q < \frac{\epsilon}{2\bar{\nu}(S_0)}$$

independently of τ. Now for Q not in $S_0(\delta, P)$ the quantity $\left| \dfrac{\sin\theta}{r^2} - \dfrac{\sin\theta'}{(r')^2} \right|$

is bounded for each τ. Let $K_1(\tau)$ be the least upper bound of this quantity. We have then

$$\int_{S_0} | U_3(M) - U_3(M') | \, dS_Q < K_1(\tau) m(S_0) \bar{\nu}(S_0) + \epsilon/2$$

for each τ. But $K_1(\tau)$ obviously approaches zero uniformly with τ. Hence the theorem is proved.

THEOREM 2. 3. *The integrals*

$$\int_S | U(M) | \, dS_M ; \quad \int_S | U_1(M) | \, dS_M ; \quad \int_{S_0} | U(M) | \, dS_Q ; \quad \int_{S_0} | U_1(M) | \, dS_Q$$

remain bounded as τ approaches zero. The same holds if $U(M')$, $U_1(M')$, are substituted for $U(M)$, $U_1(M)$, respectively.

It is sufficient to prove the theorem for the first two integrals. In fact, we need only to prove the theorem for the second integral, for

$$\begin{aligned} \int_S \big| \, | U(M) | - | U_1(M) | \, \big| \, dS_M &\leq \int_S | U(M) - U_1(M) | \, dS_M \\ &\leq \int_{S_0} | d\nu(e_P) | \int_S \left| \frac{\cos \phi - \cos \theta}{r^2} \right| dS_M \end{aligned}$$

which is bounded by Lemma 2. Consider then

$$\int_S | U_1(M) | \, dS_M \leq \int_S dS_M \int_{S_0} \left| \frac{\cos \theta}{r^2} \right| d\nu(e_P).$$

Supposing for the moment that $\theta = \theta_M$, we have

$$\int_S \left| \frac{\cos \theta_M}{r^2} \right| dS_M = \int_S \left| \frac{\cos \sphericalangle \, (MP, n_M)}{r^2} \right| dS_M.$$

But S is a surface of "class Γ."[3] Hence a positive constant Γ exists so that $\int_S \left| \frac{\cos \sphericalangle \, (MP, n_M)}{r^2} \right| dS_M < \Gamma$. Hence we may interchange the order of integration above, obtaining

$$\int_S | U_1(M) | \, dS_M \leq \int_{S_0} | d\nu(e_P) | \int_S \frac{| \cos \theta_M |}{r^2} dS_M < \Gamma \bar{\nu}(S_0).$$

Therefore $\int_S | U_1(M) | \, dS_M$ is bounded for $\theta = \theta_M$. Moreover, for the pur-

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[3] (A), p. 494.

poses of this theorem it is immaterial whether we use θ_M or θ_Q for θ, since $\int_S \left| \frac{\cos\theta_M - \cos\theta_Q}{r^2} \right| dS_M$ is bounded, by Lemma 2.

For the second part of the theorem, we may write

$$|U(M')| \leq |U(M') + U(M)| + |U(M)|;$$
$$|U_1(M)| \leq |U(M')| + |U_1(M') - U(M')|$$

We have only to show, that

$$\int_S |U(M') + U(M)|\, dS_M \quad \text{and} \quad \int_S |U_1(M') - U(M')|\, dS_M$$

are bounded. We have

$$\int_S |U_1(M') + U(M)|\, dS_M \leqq \int_{S_0} |d\nu(e_P)| \int_S \left| \frac{\cos\theta}{r^2} + \frac{\cos\theta'}{(r')^2} \right| dS_M.$$

The inner integral on the right is bounded by Lemma 2. Hence the iterated integral is bounded. Similarly it may be shown that

$$\int_S |U_1(M') - U(M')|\, dS_M$$

is bounded.

Let M, N on the surface S correspond to Q, P respectively on S_0.

Lemma 3. *The integrals*

$$\int_{E_S} \frac{\cos \sphericalangle (MN, n_N)}{\overline{MN}^2} dS_M \quad \textit{and} \quad \int_{E_S} \frac{\cos \sphericalangle (MN, n_M)}{\overline{MN}^2} dS_M$$

converge uniformly to the integrals

$$\int_{E_{S_0}} \frac{\cos \sphericalangle (QP, n_P)}{\overline{QP}^2} dS_Q \quad \textit{and} \quad \int_{E_{S_0}} \frac{\cos \sphericalangle (QP, n_Q)}{\overline{QP}^2} dS_Q$$

respectively as τ approaches zero.

We may write

$$\left| \int_{E_{S_0}} \frac{\cos \sphericalangle (QP, n_P)}{\overline{QP}^2} dS_Q - \int_{E_S} \frac{\cos \sphericalangle (MN, n_N)}{\overline{MN}^2} dS_M \right| \leq J_1 + J_2$$

where

$$J_1 = \left| \int\limits_{E_{S_0} - E_{S_0} \cdot S_0(\delta, P)} \frac{\cos \sphericalangle (QP, n_P)}{\overline{QP}^2} dS_Q - \int\limits_{E_S - E_S \cdot S(\delta, P)} \frac{\cos \sphericalangle (MN, n_N)}{\overline{MN}^2} dS_M \right|;$$

$$J_2 = \left| \int\limits_{E_{S_0} \cdot S_0(\delta, P)} \frac{\cos \sphericalangle (QP, n_P)}{\overline{QP}^2} dS_Q - \int\limits_{E_S \cdot S(\delta, P)} \frac{\cos \sphericalangle (MN, n_N)}{\overline{MN}^2} dS_M \right|.$$

Since these are surfaces of class Γ, the two integrals composing J_2 exist and approach zero uniformly with δ on account of the absolute continuity of the integral as a function of the set over which the integration is taken. Given $\epsilon > 0$ we may fix δ so that J_2 is less than $\epsilon/2$. Now let τ be less than δ so that N is in $S(\delta, P)$. The integrand of the second integral of J_1 converges uniformly to that of the first as τ approaches zero. Also dS_Q/dS_M approaches one uniformly as τ approaches zero. Hence for τ_1 sufficiently small we have $J_1 < \epsilon/2$ for all M, N, Q, P and all $\tau \leq \tau_1$, which proves the theorem. The other two integrals are treated similarly.

3. *On the solution of integral equations.* Let S_0 be a surface of the kind described in the preceding section and let $\{S\}$ be a normal family of surfaces inside or outside S_0 and including S_0. Let w be a regular closed curve [4] on S_0 and let σ be one part of S_0 enclosed by w. Denote by $q(P, w)$ the symmetric surface density of σ at the point P; i. e.

$$\begin{aligned} q(P, w) &= 1 \quad \text{for } P \text{ inside } w, \\ &= 0 \quad \text{for } P \text{ outside } w, \\ &= \psi/2\pi \text{ for } P \text{ on } w, \text{ where } \psi \text{ is the angle} \end{aligned}$$

between the forward and backward tangents to w at P. Define the function of regular curves with regular discontinuities

$$\nu(w) = \int_{S_0} q(P, w)\, d\nu(e_P) \tag{3.1}$$

Let $S(\tau)$ be a member of the family the least upper bound of whose normal distances from S_0 is τ, and let $w(\tau)$, $\sigma(\tau)$ be the sets of points on $S(\tau)$ corresponding to w, σ respectively on S_0. Write

$$v(M) = \int_{S_0} \frac{1}{MP}\, d\nu(e_P)\,; \tag{3.2}$$

$$U(\tau, w) = \int_{\sigma(\tau)} U(M)\, dS_M\,; \tag{3.3}$$

$$V(\tau, w) = \int_{\sigma(\tau)} \frac{dv}{dn_M}\, dS_M = \int_{\sigma(\tau)} U_1(M)\, dS_M, \quad [\theta = \theta_M]. \tag{3.4}$$

Denoting approach to S_0 from T^+ and T^- by $\lim\limits_{\tau \to 0^+}$ and $\lim\limits_{\tau \to 0^-}$ respectively, we have [5]

$$\lim_{\tau \to 0^\pm} U(\tau, w) = \mp 2\pi\nu(w) + \int_{S_0} d\nu(e_P) \int_{\sigma} \frac{\cos \sphericalangle (QP, n_P)}{\overline{QP}^2}\, dS_Q. \tag{3.5}$$

[4] (A), p. 498.

[5] (A), p. 502.

$$(3.6) \qquad \lim_{\tau \to 0^\pm} V(\tau, w) = \mp 2\pi\nu(w) + \int_{S_0} d\nu(e_P) \int_\sigma \frac{\cos \sphericalangle (QP, n_Q)}{\overline{QP}^2} dS_Q.$$

If $F(w)$ is a given completely additive function of regular curves on S_0 with regular discontinuities, then in order to obtain the mass function $\nu(w)$ determining $F(w)$ by means of the relation $F(w) = \lim_{\tau \to 0^+} \int_{\sigma(\tau)} U(M) dS_M$, where $U(M)$ is given by (I) for M in T^+, we have the equation

$$F(w) = -2\pi\nu(w) + \int_{S_0} d\nu(w_P) \int_\sigma \frac{\cos \sphericalangle (QP, n_P)}{\overline{QP}^2} dS_Q.$$

Putting

$$K(Q,P) = \frac{1}{2\pi} \frac{\cos \sphericalangle (QP, n_P)}{\overline{QP}^2} \qquad \Phi(w) = -\frac{1}{2\pi} F(w)$$

we obtain the equation

$$(3.7) \qquad \nu(w) = \Phi(w) + \int_{S_0} d\nu(w_P) \int_\sigma K(Q,P) dS_Q.$$

THEOREM 3.1. *The equation* (3.7) *has the solution*

$$(3.8) \qquad \nu(w) = \Phi(w) - \int_{S_0} d\Phi(w_P) \int_\sigma k(Q,P) dS_Q$$

where $k(Q,P)$ *is the resolvent kernel for the Fredholm equation of the form*

$$(3.9) \qquad h(Q) = f(Q) + \int_{S_0} K(Q,P) h(P) dS_P,$$

and where $f(Q)$ *is given by* (3.12) *below. The function* $\nu(w)$ *is a completely additive function of regular curves on* S_0 *with regular discontinuities and is uniquely determined.*

The function $k(Q,P)$ is seen by its explicit expression, given by (3.17) below, to be continuous except when $Q = P$. Moreover, it is integrable with respect to Q and P over S_0, as follows incidentally in the proof of Lemma 4 below.

We show first that if $\nu(w)$ is a completely additive function of regular curves on S_0 with regular discontinuities and is a solution of (3.7) then $\nu(w)$ satisfies (3.8). Put

$$(3.10) \qquad \nu(w) = \Phi(w) + H(w).$$

It follows immediately from (3.7) that $H(w)$ is absolutely continuous. Hence the derivative of $H(w)$ exists almost everywhere and we may write

(3.11) $$H(w) = \int_{\sigma} h(R)\, dS_R,$$

where $h(R)$ is the derivative almost everywhere of $H(w)$. Substituting (3.10) in (3.7) we have

$$H(w) = \int_{S_0} d\Phi(w_P) \int_{\sigma} K(Q,P)\, dS_Q + \int_{S_0} dH(w_P) \int_{\sigma} K(Q,P)\, dS_Q.$$

Hence

$$h(Q) = \int_{S_0} K(Q,P)\, d\Phi(w_P) + \int_{S_0} K(Q,P) h(P)\, dS_P.$$

Putting

(3.12) $$f(Q) = \int_{S_0} K(Q,P)\, d\Phi(w_P)$$

we obtain the equation (3.9). But (3.9) has the unique [6] summable solution $h(Q)$ given by

(3.13) $$h(Q) = f(Q) - \int_{S_0} k(Q,R) f(R)\, dS_R.$$

where [7]

(3.14) $$\begin{aligned} k(Q,P) + K(Q,P) &= \int_{S_0} k(Q,R) K(R,P)\, dS_R \\ &= \int_{S_0} K(Q,R) k(R,P)\, dS_R. \end{aligned}$$

Substituting (3.13) in (3.11) we have

$$\begin{aligned} H(w) = &\int_{\sigma} dS_Q \int_{S_0} K(Q,P)\, d\Phi(w_P) \\ &- \int_{\sigma} dS_Q \int_{S_0} k(Q,R)\, dS_R \int_{S_0} K(R,P)\, d\Phi(w_P). \end{aligned}$$

By the first part of the lemma proved below we may change the order of integration to reduce this last equation to the form

$$H(w) = \int_{S_0} d\Phi(w_P) \int_{\sigma} dS_Q \{K(Q,P) - \int_{S_0} K(R,P) k(Q,R)\, dS_R\}.$$

Applying (3.14) we now have

$$H(w) = -\int_{S_0} d\Phi(w_P) \int_{\sigma} k(Q,P)\, dS_Q.$$

Substituting in (3.10), we obtain the equation (3.8). Similarly it may be

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[6] (A), p. 507.

[7] See Appendix 1.

shown that $\nu(w)$ given by (3.8) is a completely additive function of curves and satisfies (3.7), which completes the proof of the theorem.

LEMMA 4. *The integrals*

$$\int_{S_0} |k(Q,P)|\, dS_Q \quad \text{and} \quad \int_{S_0} |k(Q,P)|\, dS_P$$

are bounded for all Q, P on S_0. If M, N on $S(\tau)$ correspond respectively to Q, P on S_0, then

$$\int_{S(\tau)} |k(M,N)|\, dS_M \quad \text{and} \quad \int_{S(\tau)} |k(M,N)|\, dS_N$$

are bounded uniformly as τ approaches zero.

We define the iterated kernels $K_i(Q,P)$ by the relation

$$K_{i+1}(Q,P) = \int_{S_0} K(Q,R) K_i(R,P)\, dS_R$$

where $K_0(R,P) = K(R,P)$. The kernel $K(Q,P)$ is not bounded. However $K_1(Q,P)$ is bounded and continuous except when $Q = P$ and $K_2(Q,P)$ is continuous [8] in Q and P. It may readily be shown that $K_1(M,N)$ is bounded uniformly as τ approaches zero.[9] Making use of this fact in conjunction with Lemma 3, we see that $K_i(M,N)$, $(i \geq 2)$, changes continuously as τ approaches zero, the continuity being uniform in M, N. A proof of this last statement follows.

First, we prove the statement for $i = 2$. Let R on S_0 and G on $S(\tau)$ be corresponding points. Then for any M, N, Q, P

$$\left| \int_{S_0} K(Q,R) K_1(R,P)\, dS_R - \int_{S(\tau)} K(M,G) K_1(G,N)\, dS_G \right| \leq J_1 + J_2$$

where

$$J_1 = \left| \int_{S_0 - S_0(\delta,Q) - S_0(\delta,P)} K(Q,R) K_1(R,P)\, dS_R - \int_{S(\tau) - S(\delta,Q) - S(\delta,P)} K(M,G) K_1(G,N)\, dS_G \right| ;$$

$$J_2 = \left| \int_{S_0(\delta,Q) + S_0(\delta,P)} K(Q,R) K_1(R,P)\, dS_R - \int_{S(\delta,Q) + S(\delta,P)} K(M,G) K_1(G,N)\, dS_G \right| .$$

Due to the uniform boundedness of $K_1(M,N)$ we may fix δ so that J_2 is as small as we like. Then we let τ be less than δ so that M and N are in $S(\delta, Q)$

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[8] (A), p. 507. See also Kellogg, *Potential Theory*, p. 301.

[9] From the nature of the proof that $K_1(Q,P)$ is bounded, using the proof suggested in the footnote given on p. 507 of (A), the uniform boundedness of $K_1(M,N)$ is seen immediately. See Appendix 2.

and $S(\delta, P)$ respectively. Now the integrand of the second integral in J_1 converges uniformly to that of the first as τ approaches zero and dS_R/dS_G approaches one uniformly, which proves the statement. The proof for $i > 2$ is an obvious application of the method of mathematical induction.

Replacing $K(Q, P)$ by $\lambda K(Q, P)$ and substituting (3. 9) into itself twice we obtain the equivalent [10] equation

$$h(Q) = f_2(Q) + \lambda^3 \int_{S_0} K_2(Q, P) h(P) dS_P \tag{3.15}$$

where

$$f_2(Q) = f(Q) + \lambda \int_{S_0} K(Q, P) f_1(P) dS_P$$

and

$$f_1(Q) = f(Q) + \lambda \int_{S_0} K(Q, P) f(P) dS_P.$$

The solution of (3. 15) may be written

$$h(Q) = f_2(Q) - \lambda^3 \int_{S_0} k_2(Q, R; \lambda^3) f_2(R) dS_R \tag{3.16}$$

where $k_2(Q, R; \lambda^3)$ is continuous in Q and R, providing λ^3 is not a characteristic value for $k_2(Q, R; \lambda^3)$. Denoting by $k(Q, P; \lambda)$ the resolvent kernel for (3. 9) where $K(Q, P)$ has been replaced by $\lambda K(Q, P)$ we have, from (3. 13),

$$\begin{aligned} k(Q, R; \lambda) = &- K(Q, R) - \lambda K_1(Q, R) + \lambda^2 k_2(Q, R; \lambda^3) \\ &+ \lambda^3 \int_{S_0} k_2(Q, P; \lambda^3) K(P, R) dS_P + \lambda^4 \int_{S_0} k_2(Q, P; \lambda^3) K_1(P, R) dS_P. \end{aligned} \tag{3.17}$$

Since $\lambda = 1$ is not a characteristic value for $K(Q, P)$ and the characteristic values of $K(Q, P)$ are real, $\lambda^3 = 1$ is not a characteristic value for $K_2(Q, P)$, and $k_2(Q, R; 1)$ is a continuous function of Q and R. From the known expressions for $k_2(Q, P; \lambda^3)$ in terms of the iterated kernels of index at least as great as two it is evident that $k_2(M, N; 1)$ converges uniformly to $k_2(Q, P; 1)$ as τ approaches zero. That $k(Q, P)$ and $k(M, N)$ have the properties stated in the lemma follows from (3. 17) and the fact that $K(Q, P)$ and $K(M, N)$ have these properties.

From the solution (3. 8) of the equation (3. 7) we may obtain an explicit formula for the potential given by (I) in terms of $F(w)$. Substituting (3. 8) in (I), we have

$$U(M) = \frac{1}{4\pi} \int_{S_0} g_n(M, P) dF(w_P) = \frac{1}{4\pi} \int_{S_0} g_n(M, P) dF(e_P) \tag{3.18}$$

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[10] That (3. 9) and (3. 15) are equivalent if λ is not a characteristic value is shown in Goursat, *Cours d'Analyse Mathématique*, vol. 3, pp. 355-356.

where

$$(3.19)\quad g_n(M,P)=2\int_{S_0}k(R,P)\frac{\cos\sphericalangle(MR,n_R)}{\overline{MR}^2}dS_R-2\frac{\cos\sphericalangle(MP,n_P)}{\overline{MP}^2}.$$

The function of point sets $\nu(e)$ and the function of regular curves $\nu(w)$ are connected by the relations

$$(3.20)\quad \begin{cases} \nu(w)=\int_{S_0}q(P,w)d\nu(e_P)=\int_{S_0}q(P,w)d\nu(w_P) \\ \nu(e)=\int_{S_0}\phi(P,e)d\nu(w_P)=\int_{S_0}\phi(P,e)d\nu(e_P) \end{cases}$$

where $\phi(P,e)$ is the characteristic function of the set e. Similar relations exist [11] between $F(e)$ and $F(w)$, and also between the positive and negative variation functions $F_\pm(e)$, $F_\pm(w)$ and $\nu_\pm(e)$, $\nu_\pm(w)$. Since the function of point sets is completely additive, the corresponding function of regular curves is completely additive and conversely.

THEOREM 3.2. *If $U(M)$ is given by (I) for M in T^+, a necessary and sufficient condition that $\nu(e)$ be absolutely continuous is that $F(e)$ be absolutely continuous. Further, if in this situation $\nu'(P)$ is continuous, then $F'(P)$ is continuous and conversely, where $\nu'(P)$ and $F'(P)$ are the derivatives respectively of $\nu(e)$ and $F(e)$.*

From the equations corresponding to (3.20) for the positive and negative variation functions for $\nu(e)$ and $F(e)$ and $\nu(w)$ and $F(w)$ we have, denoting by $\bar\sigma$ the set of points composing w,

$$\nu_\pm(\sigma)\leq\nu_\pm(w)\leq\nu_\pm(\sigma+\bar\sigma)$$
$$F_\pm(\sigma)\leq F_\pm(w)\leq F_\pm(\sigma+\bar\sigma).$$

Suppose $\nu(e)$ is absolutely continuous. Then $\nu_\pm(e)$ are absolutely continuous. Since $\nu_\pm(e)$ are also additive, we have

$$\nu_\pm(\sigma+\bar\sigma)=\nu_\pm(\sigma)$$

since $\bar\sigma$ is of zero superficial measures. Hence $\nu_\pm(w)=\nu_\pm(\sigma)$. From (3.7) we may write

$$F_\pm(\sigma)\leq F_\pm(w)\leq 2\pi\nu_+(\sigma)+2\pi\nu_-(\sigma)+\int_{S_0}|d\nu(e_P)|\int_\sigma\frac{|\cos\sphericalangle(QP,n_P)|}{\overline{QP}^2}dS_Q$$

[11] See Bray and Evans, "A class of functions harmonic within the sphere," *American Journal of Mathematics*, vol. 49 (1927), pp. 158-159. The proofs given for functions of segments on the sphere apply equally well to bounded additive functions of regular curves on S_0.

where each term on the right is absolutely continuous. Therefore $F_{\pm}(e)$, and hence $F(e)$, are absolutely continuous. For the sufficiency of the condition stated in the theorem we use the same method of proof, but use equation (3. 8) instead of (3. 7).

We note that if $U(M)$ is given by (I) where $\nu(e) = \int_e \nu'(P)dS_P$ and if $\nu'(P)$ is continuous on S_0, then $U(M)$ takes on continuously its boundary values $F'(P)$. In fact

$$U(M) = \int_{S_0} \frac{\cos \sphericalangle (MP, n_P)}{\overline{MP}^2} \nu'(P)dS_P$$

and the statement follows as in the case of regular surfaces.[12]

THEOREM 3. 3. *If $U(M)$ is given by (I), it may be written in T^+ in the form* (3. 18) *where $g_n(M,P)$ is continuous in P and harmonic in M, for M in T^+, and is not negative; and*

$$F(w) = \lim_{\tau \to 0^+} U(\tau, w) = \lim_{\tau \to 0^+} \int_{\sigma(\tau)} U(M)dS_M$$

where M is on the surface $S(\tau)$ of the normal family inside S_0.

That $g_n(M,P)$ is harmonic in M and continuous in P for M in T^+ follows immediately[13] from the equation (3. 19). We have only to show that $g_n(M,P)$ is not negative for M in T^+ and P on S_0. Suppose the contrary. Then for some point M_1 in T^+ and some point P_1 of S_0, $g_n(M,P)$ is negative. From the continuity of $g_n(M,P)$ in P there is a neighborhood ω_δ on S_0 containing P_1 such that $g_n(M_1,P) < 0$ for P in ω_δ. Suppose $F(w) = \int_\sigma F'(P)dS_P$ where $F'(P)$ is continuous on S_0, positive for P in ω_δ, and zero otherwise. Then

$$U(M_1) = \frac{1}{4\pi} \int_{\omega_\delta} g_n(M_1,P)F'(P)dS_P < 0.$$

But $U(M)$ cannot be negative anywhere in T^+ since it is the solution of the interior Dirichlet problem for continuous not-negative boundary values. This contradiction establishes the theorem.

THEOREM 3. 4. *The function $U(M)$ given by (I) for M in T^+ is the difference of two not-negative functions harmonic in T^+.*

This theorem follows immediately from the equation (3. 18) and the

[12] Kellogg, *Potential Theory*, pp. 167-168.

[13] See Appendix 3.

preceding theorem, together with the fact that the completely additive function $F(w)$ may be expressed as the difference of two not-negative functions.

THEOREM 3.5. *If $U(M)$ is given by (I) for M in T^+, a necessary and sufficient condition that $\nu(e)$ be absolutely continuous is that the absolute continuity of the integrals $\int_{\sigma(\tau)} U(M)\,dS_M$ be uniform as τ approaches zero.*

We prove first the necessity of the condition, supposing $\nu(e)$ to be absolutely continuous. $F(e)$ is absolutely continuous by the preceding theorem. It is sufficient to consider the case where $F(e)$, and therefore $U(M)$, is not negative. But

$$F(\sigma) = \lim_{\tau\to 0^+} \int_{\sigma(\tau)} U(M)\,dS_M.$$

Since $F(\sigma)$ is an absolutely continuous function of point sets, it has a derivative $F'(P)$ almost everywhere, and

$$(3.21) \qquad \begin{cases} F(\sigma) = \int_\sigma F'(Q)\,dS_Q \\ F'(Q) = \lim\limits_{\tau\to 0^+} U(M) \end{cases}$$

almost everywhere.[14] Suppose the absolute continuity of the integrals $\int_{\sigma(\tau)} U(M)\,dS_M$ is not uniform. Then there exists a denumerable sequence $\{S_i\}$ of surfaces of the family for which the absolute continuity of the integrals $\int_{\sigma_i} U(M)\,dS_M$ is not uniform, where σ_i denotes the region on S_i corresponding to σ on S_0. By the property (c) of normal families the sequence $\{S_i\}$ converges to some surface S^* of the family. We denote by $U_i(M^*)$ the function $U(M)\,dS_M/dS_{M^*}$ where M^* on S^* corresponds to M on S_i. Hence $\int_{\sigma_i} U(M)\,dS_M = \int_{\sigma^*} U_i(M^*)\,dS_{M^*}$, where σ^* on S^* corresponds to σ_i on S_i. By De la Vallée Poussin's converse to Vitali's Theorem, either $\{\lim\limits_{i\to\infty} U_i(M^*)\}$

[14] Due to the absolute continuity of F and ν, we may compute the derivatives of both members of (3.7), obtaining

$$\nu'(Q) = -(1/2\pi)F'(Q) + \int_{S_0} K(Q,P)\nu'(P)\,dS_P$$

where $\nu'(Q)$ and $F'(Q)$ are the derivatives almost everywhere of $\nu(w)$ and $F(w)$. Transposing, we obtain

$$F'(Q) = -2\pi\nu'(Q) + 2\pi \int_{S_0} K(Q,P)\nu'(P)\,dS_P = \lim_{\substack{\tau\to 0^+ \\ (M\to Q)}} U(M),$$

as is shown in (A), p. 511.

is not summable or $\lim_{i\to\infty}\int_{\sigma^*} U_i(M^*)\,dS_{M^*}$ is not equal to $\int_{\sigma^*}\{\lim_{i\to\infty} U_i(M^*)\}dS_{M^*}$. If S^* is S_0, neither of these conditions is possible as is shown by (3.21). But if S^* is not S_0, then $U(M)$ is continuous, and again neither condition can subsist.

To prove the sufficiency, we note that if the absolute continuity of the integrals $\int_{\sigma(\tau)} U(M)\,dS_M$ is uniform, then $F'(Q)$ is summable by Vitali's Theorem, and $\lim_{\tau\to 0^+}\int_{\sigma(\tau)} U(M)\,dS_M = \int_{\sigma} F'(Q)\,dS_Q$. Hence $F(e)$ is absolutely continuous and therefore $\nu(e)$ is absolutely continuous by Theorem 3.2.

4. *Necessary and sufficient conditions for representation as potentials.*

THEOREM 4.1. *A necessary and sufficient condition that $U(M)$ harmonic in T^+ be representable in the form (I) is that $\int_{S(\tau)} |U(M)|\,dS_M$ remain bounded over a normal family $\{S(\tau)\}$ within S_0 as τ approaches zero.*

The necessity of the condition is given by Theorem 2.3. To prove the sufficiency we note that by hypothesis the functions of point sets $F(\sigma(\tau))$, given by

$$F(\sigma(\tau)) = \int_{\sigma(\tau)} U(M)\,dS_M,$$

and therefore the corresponding functions of regular curves $F(w(\tau))$ defined on the surfaces $S(\tau)$, are of uniformly bounded total variation. Moreover $U(M)$ is uniformly continuous inside and on $S(\tau)$. From the equations corresponding to (3.8) formed for the surfaces of the family it follows that the functions $\nu(w(\tau))$ are of uniformly bounded total variation, and by Theorem 3.2 we may write $\nu(w(\tau)) = \int_{\sigma(\tau)} \nu'(M)\,dS_M$ where $\nu'(M)$ is continuous on $S(\tau)$. Consequently we may pick out a subsequence $\{S_k\}$ of surfaces of the family for which the functions $\nu_k(w')$ converge in the weak sense[15] to a completely additive function $\nu(w)$ defined on S_0, where the $\nu_k(w')$ represent the solutions of the equations corresponding to (3.7) for the surfaces S_k. Denoting by P_k a variable point on S_k, we have, for k sufficiently large,

$$U(M) = \int_{S_k} \frac{\cos \sphericalangle\,(MP_k, n_{P_k})}{\overline{MP}_k^{\,2}}\, d\nu_k(w'_{P_k}).$$

[15] J. Radon, "Über die Randwertaufgaben beim Logarithmischen Potential," *Wiener Akademie Sitzungsberichte*, vol. 128 (1919), IIa, p. 1153. The methods employed by Radon for a certain class of curves may be extended readily to apply to normal families of surfaces. See Appendix 4.

In fact, the $\nu_k(e)$ is the integral of a continuous $\nu'(P_k)$ and the right-hand member is harmonic within S_k and takes on continuously the boundary values $U(M_k)$ as M tends to M_k on S_k. But there can be only one such function. Consequently that function is $U(M)$. From the weak convergence of the functions $\nu_k(w')$ to $\nu(w)$ defined on S_0, we now have

$$U(M) = \int_{S_0} \frac{\cos \sphericalangle (MP, n_P)}{\overline{MP}^2}\, d\nu(w_P) = \int_{S_0} \frac{\cos \sphericalangle (MP, n_P)}{\overline{MP}^2}\, d\nu(e_P)$$

which is what was to be shown.

Lemma 5. *The function $g_n(M,P)$ given by* (3.18) *is positive for M in T^+ and P on S_0.*

By Theorem 3. 3, $g_n(M,P)$ is not-negative. Suppose there exists a point M_1 of T^+ and a point P_1 of S_0 such that $g_n(M_1,P_1)$ is zero. Then $g_n(M,P_1)$ is identically zero, since $g_n(M,P)$ is harmonic in M and not-negative. For any given completely additive function of regular curves with regular discontinuities, $F(w)$, we may solve the equation (3. 7) for $\nu(w)$ so that $U(M)$ is given by (3. 18) and

$$F(w) = \lim_{\tau\to 0^+} \int_{\sigma(\tau)} U(M)\, dS_M.$$

Define $F(e)$ so that

$$F(e) = 1 \text{ if } e \text{ contains } P_1.$$
$$F(e) = 0 \text{ otherwise.}$$

From (3. 17) we now have $U(M) \equiv 0$. Hence

$$F(w) = \lim_{\tau\to 0^+} \int_{\sigma(\tau)} U(M)\, dS_M = 0$$

for all w, and therefore $F(e) = 0$ for all e, which contradicts the assumption that $F(e) = 1$ for all e containing P_1.

Theorem 4. 2. *A necessary and sufficient condition*[16] *that $U(M)$ harmonic in T^+ be given by (I) is that $U(M)$ be the difference of two not-negative functions harmonic in T^+.*

The necessity of the condition is given by Theorem 3. 4. For the suffi-

[16] De la Vallée Poussin (*loc. cit.*, p. 199) proves that a necessary and sufficient condition that the harmonic function U defined in T^+ be the difference of two positive harmonic functions is that $\int_S |U|\, dS$ be bounded over a normal family of surfaces in T^+ and arbitrarily near S_0. This theorem follows directly from Theorems 4. 1 and 4. 2.

ciency we need only prove the theorem for $U(M)$ not-negative and harmonic in T^+. If $U(M)$ vanishes anywhere in T^+, then it vanishes identically, and the theorem is trivial. Hence, we shall suppose that $U(M)$ is positive. Consider a fixed point $\bar{M}$ of T^+ and a normal family $S(\tau)$ of surfaces inside S_0, where $\tau_1 \geq \tau \geq 0$ and τ_1 is small enough so that $\bar{M}$ is inside all the surfaces. We form the functions $g_n(\bar{M}, N)$ corresponding to $g_n(\bar{M}, P)$ where N is a point of $S(\tau)$. From the form of $g_n(\bar{M}, P)$ given by (3.19) and from Lemma 4 it is easily seen that $g_n(\bar{M}, N)$ converges uniformly[17] to $g_n(\bar{M}, P)$ as N on $S(\tau)$ approaches P on S_0 along n_P and that the convergence is uniform in τ and P. There exists a number $\epsilon > 0$ such that

$$g_n(\bar{M}, N) \geq \epsilon$$

for all N of $S(\tau)$ and all τ; for, supposing the contrary, we can find a sequence of points N_i on surfaces of the family, having a limit point $\bar{N}$ on one of the surfaces such that $g_n(\bar{M}, \bar{N})$ equals zero. This is impossible by Lemma 5. Defining the functions $F(w(\tau))$ on $S(\tau)$ by the relation

$$F(w(\tau)) = \int_{S(\tau)} U(M)\, dS_M$$

we have for every τ in $\tau_1 \geq \tau \geq 0$

$$U(\bar{M}) = \frac{1}{4\pi} \int_{S(\tau)} g_n(\bar{M}, N)\, dF(w(\tau)_N).$$

In fact $F(\sigma(\tau))$ is the integral of a continuous function $F'(N)$, and $F'(N) = \lim\limits_{M \to N} U(M)$. That $U(\bar{M})$ may be written in this form follows from Theorem 3.3 and the fact that inside $S(\tau)$, $U(\bar{M})$ is given by

$$U(M) = \int_{S(\tau)} \frac{\cos \measuredangle\, (MN, n_N)}{\overline{MN}^2}\, dv(w(\tau)_N)$$

as was noted in the proof of Theorem 4.1. Now $F(w(\tau))$ is positive since $U(M)$ is positive. Hence

$$U(\bar{M}) \geq \frac{\epsilon}{4\pi} F(S(\tau)).$$

Therefore the functions $F(S(\tau))$ remain bounded uniformly over the normal family inside S_0. The theorem now follows immediately from Theorem 4.1.

THEOREM 4.3. *A necessary and sufficient condition that $v(M)$ harmonic in T^-, except at infinity, be representable in T^- by* (3.2) *is that*

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[17] Appendix 5.

(a) *$v(M)$ approach zero continuously at infinity and*

(b) $\int_{S(\tau)} |U_1(M)|\, dS_M = \int_{S(\tau)} \left| \frac{dv}{dn_M} \right| dS_M$ *remain bounded over a normal family $S(\tau)$ outside S_0 as τ approaches zero.*

The necessity of (a) is immediate. The necessity of (b) is given by Theorem 2.3. Before proving the sufficiency we consider (3.6) in the form

$$G(w) = 2\pi\nu(w) + \int_{S_0} d\nu(e_P) \int_\sigma \frac{\cos \sphericalangle\, (QP, n_Q)}{\overline{QP}^2}\, dS_Q;$$

or

$$\nu(w) = \Psi(w) + \int_{S_0} d\nu(e_P) \int_\sigma K(P,Q)\, dS_Q \tag{4.1}$$

where

$$\Psi(w) = \frac{1}{2\pi}\, G(w) = \lim_{\tau \to 0^-} \frac{1}{2\pi} \int_{\sigma(\tau)} \frac{dv}{dn_M}\, dS_M.$$

We may solve this equation in a manner similar to the one followed in solving (3.7). We may write the solution in the form

$$\nu(w) = \Psi(w) - \int_{S_0} d\Psi(w_P) \int_\sigma k(P,R)\, dS_R. \tag{4.2}$$

Given $v(M)$, in order to prove the sufficiency of the conditions (a) and (b), we form the functions $\Psi(w(\tau)) = \int_{\sigma(\tau)} U_1(M)\, dS_M$. By hypothesis, these functions are of uniformly bounded total variation. From the equations corresponding to (4.2) formed for the surfaces $S(\tau)$ it is seen that the functions $\nu(w(\tau))$ have the same property. Hence, as in the proof of Theorem 4.1, we may pick out subsequences $\nu_k(w')$ defined on surfaces S_k of the family outside S_0 converging in the weak sense to a completely additive function $\nu(w)$ defined on S_0. From (a) it follows that $v(M)$ is regular [18] at infinity. Since $U_1(M)$ is continuous outside S_k and takes on continuous boundary values on S_k, it follows from (4.2) that $\nu_k(e)$ is the integral of a continuous function. Hence for k sufficiently large, we may write as for regular surfaces [19]

[18] It follows immediately from a theorem given by Poincaré in his *Théorie du Potentiel Newtonien*, p. 210, that we may write

$$v(M) = v(x,y,z) = K + H_0/R + H_1/R^3 + \cdots + H_n/R^{2n+1} + \cdots$$

valid outside a sphere of sufficiently large radius; where K is a constant; H_i is a spherical harmonic of degree i in x, y, z; and $R = \sqrt{x^2 + y^2 + z^2}$. We note at once that $K = 0$, and hence $v(M)$ approaches zero canonically at infinity.

[19] Kellogg, *Potential Theory*, p. 311.

$$v(M) = \int_{S_k} \frac{1}{MP_k} d\nu_k(w'_{P_k})$$

for the right-hand member is harmonic outside S_k and is regular at infinity and its normal derivative takes on continuously [20] the boundary values $U_1(M_k)$ as M tends to M_k on S_k. But there can be only one such function.[21] Consequently that function is $v(M)$. From the weak convergence, we now have

$$v(M) = \int_{S_0} \frac{1}{MP} d\nu(w_P) = \int_{S_0} \frac{1}{MP} d\nu(e_P),$$

which was to be shown.

THEOREM 4. 4. *A necessary and sufficient condition that $v(M)$ harmonic in T^+ be representable in T^+ in the form (3. 2) is that $\int_{S(\tau)} | U_1(M) | dS_M$ remain bounded over a normal family $S(\tau)$ inside S_0 as τ approaches zero.*

The necessity of the condition is given by Theorem 2. 3. Incidentally, for $v(M)$ harmonic in T^+ we have $G(S(\tau)) = \int_{S(\tau)} \frac{dv}{dn_M} dS_M = 0$. Before proving the sufficiency we first consider the equation corresponding to (3. 6). This equation is

$$\nu(w) = \Psi(w) - \int_{S_0} d\nu(w_P) \int_{\sigma} K(P, Q) dS_Q \tag{4. 3}$$

where

$$\Psi(w) = -\frac{1}{2\pi} \lim_{\tau \to 0^+} \int_{\sigma(\tau)} \frac{dv}{dn_M} dS_M.$$

The value $\lambda = -1$ is a characteristic value for the kernels $\lambda K(Q, P)$ and $\lambda K(P, Q)$ and the equation (4. 3) can be satisfied if and only if $\Psi(S_0)$ is zero. The mass function $\nu(w)$ is then determined except for an arbitrary additive term of the form $C \int_{\sigma} \phi_2(P) dS_P$, where C is a constant and $\phi_2(P)$ is a solution of the homogeneous equation [22]

$$\phi(P) = -\int_{S_0} \phi(Q) K(Q, P) dS_Q.$$

From the resolvent kernel $k(P, Q)$ we can find a particular solution of (4. 3).

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[20] *Ibid.*, p. 164.

[21] *Ibid.*, p. 213.

[22] (A), p. 505. (An error in printing occurs here. The λ should be replaced by $-\lambda$).

We first reduce (4. 3) to the Fredholm form. We put $\nu(w) = \Psi(w) - H(w)$. From (4. 3) it is seen that $H(w)$ is absolutely continuous, and hence we may write

$$H(w) = \int_\sigma h(R)\,dS_R$$

where $h(R)$ is the derivative almost everywhere of $H(w)$. Proceeding as in the proof of Theorem 4. 1 we obtain the Fredholm equation

$$h(Q) = f(Q) - \int_{S_0} K(P,Q)h(P)\,dS_P \tag{4.4}$$

where

$$f(Q) = \int_{S_0} K(P,Q)\,d\Psi(w_P).$$

We write

$$k(P,Q;\lambda) = \frac{A(P,Q)}{\lambda+1} + B(P,Q;\lambda)$$

where $A(P,Q)$ is the residue at the pole $\lambda = -1$ and is continuous.[23] A particular solution of (4. 4) is now given by

$$h(Q) = f(Q) + \int_{S_0} B(P,Q;-1)f(P)\,dS_P$$

or

$$h(Q) = f(Q) + \int_{S_0} B(P,Q;-1)f(P)\,dS_P - \int_{S_0} A(P,Q)f(P)\,dS_P \tag{4.5}$$

since

$$\int_{S_0} A(P,Q)f(P)\,dS_P = 0.^{24}$$

From (4. 5) we have

$$\begin{aligned} H(w) = &\int_\sigma dS_Q \int_{S_0} K(P,Q)\,d\Psi(w_P) \\ &+ \int_\sigma dS_Q \int_{S_0} B(R,Q;-1)\,dS_R \int_{S_0} K(P,R)\,d\Psi(w_P) \\ &- \int_\sigma dS_Q \int_{S_0} A(R,Q)\,dS_R \int_{S_0} K(P,R)\,d\Psi(w_P). \end{aligned}$$

The first part of the lemma proved below permits us to change the order of integration in this equation so that we may write

$$\begin{aligned} H(w) = \int_{S_0} d\Psi(w_P) \int_\sigma dS_Q\{K(P,Q) &+ \int_{S_0} K(P,R)B(R,Q;-1)\,dS_R \\ &- \int_{S_0} K(P,R)A(R,Q)\,dS_R\}. \end{aligned}$$

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[23] Kellogg, *Potential Theory*, p. 308.

[24] Kellogg, *loc. cit.*, p. 298.

From (3. 14) we obtain the relation

$$K(P,Q)+B(P,Q;\lambda) = \int_{S_0} K(P,R)A(R,Q)\,dS_R - \int_{S_0} K(P,R)B(R,Q;\lambda)\,dS_R.$$

Substituting in the previous equation, we now have

$$H(w) = -\int_{S_0} d\Psi(w_P)\int_{\sigma} B(P,Q;-1)\,dS_Q$$

or

$$(4.6)\qquad \nu(w) = \Psi(w) + \int_{S_0} d\Psi(w_P)\int_{\sigma} B(P,Q;-1)\,dS_Q$$

as a particular solution of (4. 3)

LEMMA 6. *The integrals*

$$\int_{S_0} |B(P,Q;-1)|\,dS_P \quad \textit{and} \quad \int_{S_0} |B(P,Q;-1)|\,dS_Q$$

are bounded for all Q, P. *The integrals*

$$\int_{S(\tau)} |B(N,M;-1)|\,dS_N \quad \textit{and} \quad \int_{S(\tau)} |B(N,M;-1)|\,dS_M$$

are bounded uniformly as τ *approaches zero.*

We may write

$$k_2(Q,P;\lambda^3) = \frac{A_2(Q,P)}{\lambda^3+1} + B_2(Q,P;\lambda^3)$$

where $A_2(Q,P)$ is continuous in Q, P and $B_2(Q,P;\lambda^3)$ is a power series in (λ^3+1) uniformly convergent in the neighborhood of $\lambda=-1$ and with coefficients which are continuous[25] in Q and P. Moreover, $B_2(M,N;-1)$ and $A_2(M,N)$ are bounded as τ approaches zero since $k_2(M,N;\lambda^3)$ changes continuously with τ. Substituting in (3. 17) and equating like powers of $(\lambda+1)$, we obtain for $\lambda=-1$

$$(4.7)\quad B(Q,R;-1) = -K(Q,R) + K_1(Q,R) - \tfrac{1}{3}A_2(Q,R) + B_2(Q,R;-1) + \tfrac{2}{3}\int_{S_0} A_2(Q,P)K(P,R)\,dS_P - \int_{S_0} A_2(Q,P)K_1(P,R)\,dS_P - \int_{S_0} B_2(Q,P;-1)K(P,R)\,dS_P + \int_{S_0} B_2(Q,P;-1)K_1(P,R)\,dS_P.$$

[25] Kellogg, *loc. cit.*, p. 294.

Since

$$A_2(Q,R) = -\int_{S_0} K_2(Q,P)A_2(P,R)\,dS_P$$
$$= -\int_{S_0} K(Q,P)A_2(P,R)\,dS_P = +\int_{S_0} K_1(Q,P)A_2(P,R)\,dS_P$$

we reduce (4.7) to the form

$$(4.8)\quad B(Q,R;-1)$$
$$= -K(Q,R) + K_1(Q,R) + B_2(Q,R;-1) - 2A_2(Q,R)$$
$$-\int_{S_0} B_2(Q,P;-1)K(P,R)\,dS_P + \int_{S_0} B_2(Q,P;-1)K_1(P,R)\,dS_P.$$

The lemma is proved by treating (4.8) in precisely the same manner as we treated (3.17) in proving Lemma 4.

We continue now with the proof of Theorem 4.4. By hypothesis the functions $\Psi(w(\tau)) = -\frac{1}{2\pi}\int_{\sigma(\tau)} \frac{dv}{dn_M}\,dS_M$ are of uniformly bounded total variation. Since $\Psi(S(\tau)) = 0$, the equations corresponding to (4.3) formed for the surfaces $S(\tau)$ may be solved for the mass functions $\bar{\nu}(w(\tau))$ defining on $S(\tau)$ the functions[26] $\Psi(w(\tau))$. The functions $\bar{\nu}(w(\tau))$, particular solutions of the equations corresponding to (4.3), given by the equations of the form (4.6) are of uniformly bounded total variation. Hence we may select a subsequence $\bar{\nu}_k(w')$ defined on the surfaces S_k and converging in the weak sense to $\bar{\nu}(w)$ defined on S_0. These functions $\bar{\nu}_k(w')$ satisfy the equations of the form (4.3), but so also do the functions

$$\nu_k(w') = \bar{\nu}_k(w') + C_k\int_{\sigma} \phi_2(P_k)\,dS_{P_k}$$
$$= \bar{\nu}_k(w') + \nu^*_k(w')$$

where w', σ', P_k on S_k correspond respectively to w, σ, P on S_0; C_k is an arbitrary constant; and $\phi_2(P_k)$ is a solution of the homogeneous equation $\phi(P_k) = -\int_{S_k} \phi(M)K(M,P_k)\,dS_M$. The potential of the form (3.2) due to the mass function $\nu^*_k(w')$ reduces inside S_k to a constant[27] K_k, where

$$K_k = \int_{S_k} \frac{1}{MP_k}\,d\nu^*_k(w'_{P_k}) = C_k\int_{S_k} \frac{1}{MP_k}\,\phi_2(P_k)\,dS_{P_k}.$$

By properly choosing the constants K_k, we have

[26] (A), p. 505.
[27] (A), p. 509 (footnote).

$$v(M) - K_k = \int_{S_k} \frac{1}{MP_k}\, d\nu_k(w'_{P_k})$$

for M inside S_k; for, as in the previous theorem, $\nu_k(w')$ is the integral of a continuous function and the normal derivative of the right-hand member takes on the same boundary values as does the normal derivative of $v(M)$ and therefore the right-hand member differs from $v(M)$ by a constant.[28] Since $v(M)$ is given and the functions $\bar{\nu}_k(w')$ converge in the weak sense to $\bar{\nu}(w)$ defined on S_0, we have for M in T^+

$$v(M) = \int_{S_0} \frac{1}{MP}\, d\bar{\nu}(w_P) + K$$

where $K = \lim\limits_{k\to\infty} K_k$. But inside T^+ we may write

$$K = B \int_{S_0} \frac{1}{MP}\, \phi_2(P)\, dS_P,$$

where B is a constant, due to the fact that

$$v_0(M) = \int_{S_0} \frac{1}{MP}\, \phi_2(P)\, dS_P \neq 0$$

(for otherwise, $v_0(M)$ being a conductor potential, we have $\phi_2(P) \equiv 0$, which is impossible since $\phi_2(P)$ is a non-zero solution of the homogeneous equation). Defining $\nu(w)$ by the relation

$$\nu(w) = \bar{\nu}(w) + B \int_{\sigma} \phi_2(P)\, dS_P$$

we now have

$$v(M) = \int_{S_0} \frac{1}{MP}\, d\nu(w_P) = \int_{S_0} \frac{1}{MP}\, d\nu(e_P)$$

which establishes the theorem.

THEOREM 4. 5. *A necessary and sufficient condition that $U(M)$ harmonic in T^-, except at infinity, be representable in T^- in the form (I) is that*

(a) *$U(M)$ approach zero continuously at infinity and* $\displaystyle\int_{\Omega} \frac{dU}{dn}\, dS = 0$,

where Ω is a sphere of radius $r > r_0$ sufficiently large, and

(b) *that $\int_{S(\tau)} |U(M)|\, dS_M$ remain bounded uniformly over a normal family $S(\tau)$ outside S_0 as τ approaches zero.*

[28] Kellogg, *Potential Theory*, p. 213.

The necessity of (a) is well known. The necessity of (b) is given by Theorem 2. 3. To prove the sufficiency we must be able to solve the integral equations corresponding to (3. 5) for the surfaces $S(\tau)$. These equations take the form

$$(4.\,9) \qquad \nu(w(\tau)) = \Phi(w(\tau)) - \int_{S_0} d\nu(e_N) \int_{\sigma(\tau)} K(M, N)\, dS_M$$

where

$$\Phi(w(\tau)) = \frac{1}{2\pi} \int_{\sigma(\tau)} U(M)\, dS_M.$$

A sufficient condition that these equations have solutions is that

$$\int_{S(\tau)} \phi_2(M)\, d\Phi(e_M) = \frac{1}{2\pi} \int_{S(\tau)} \phi_2(M)\, U(M)\, dS_M = 0$$

where $\phi_2(M)$ is a solution of the homogeneous equation

$$\phi(N) = - \int_{S(\tau)} \phi(M) K(M, N)\, dS_M.$$

But the potential of the form (3. 2) for $\nu(w(\tau)) = \int_{S(\tau)} \phi_2(M)\, dS_M$ may be expressed as a conductor potential, and hence we may take $\phi_2(M)$ as the density of the distribution producing the conductor potential, which we denote by $V(M)$. Now let Γ be a sphere of sufficiently large radius, a, and fixed center. We have

$$\int_{S(\tau)+\Gamma} U \frac{dV}{dn^-}\, dS - \int_{S(\tau)+\Gamma} V \frac{dU}{dn^-}\, dS = 0.$$

Since $U(M)$ vanishes at infinity like $1/r^2$ and dU/dn^- vanishes like $1/r^3$,[29] we may let a become infinite, obtaining

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[29] As we noted in the footnote to Theorem 4. 3, we may write

$$U(M) = U(x, y, x) = H_0/R + H_1/R^3 + \cdots + H_n/R^{2n+1} + \cdots$$

uniformly convergent outside a sphere of radius sufficiently large, where

$$R = \sqrt{x^2 + y^2 + z^2}.$$

Let Ω be a sphere of radius r large enough so that the above series converges uniformly outside and on Ω. On Ω we have

$$U(M) = H_0/r + H_1/r^3 + \cdots + H_n/r^{2n+1} + \cdots$$

and

$$\frac{dU}{dn} = - \frac{\partial U}{\partial r} = - \frac{\partial U}{\partial x}\frac{x}{r} - \frac{\partial U}{\partial y}\frac{y}{r} - \frac{\partial U}{\partial z}\frac{z}{r}.$$

But

$$\frac{\partial}{\partial r}\left(\frac{H_n}{r^{2n+1}}\right) = \frac{r^{2n+1}(H_n''/r) - (2n+1) H_n r^{2n}}{r^{4n+2}} = \frac{-H'_n(x, y, z)}{r^{2n+2}},$$

$$\int_{S(\tau)} U \frac{dV}{dn^-} dS - \int_{S(\tau)} V \frac{dU}{dn^-} dS = 0.$$

But on $S(\tau)$, dV/dn^- vanishes identically. Moreover

$$dV/dn_M^- - dV/dn_M^+ = 4\pi\phi_2(M).$$

Hence we have, since $V = 1$ on $S(\tau)$ and $\int_{S(\tau)} \frac{dU}{dn_M^-} dS_M = 0$,

$$\int_{S(\tau)} U \frac{dV}{dn_M^-} dS_M = 4\pi \int_{S(\tau)} \phi_2(M) U(M) dS_M = 0.$$

Therefore the equations (4. 9) may be solved. Consequently we may write $U(M)$ outside $S(\tau)$ in the form

$$U(M) = \int_{S(\tau)} \frac{\cos \sphericalangle (MN, n_N)}{\overline{MN}^2} d\nu(w(\tau)_N)$$

where $\nu(w(\tau))$ is the solution of (4. 9); for the right-hand member is harmonic outside $S(\tau)$, vanishes at infinity, and takes on the same values on $S(\tau)$ as $U(M)$, and is therefore identically equal to $U(M)$. The solutions of (4. 9) contain arbitrary additive constants, but we may take these constants to be zero, since they add nothing to the potentials given in the form (I) for M in T^-. To complete the proof we apply the weak convergence analysis as in the preceding theorems.

APPENDIX.

1.

We derive the equation (3. 14) by making use of the formula for $k(Q, P)$ in terms of K, K_1, and k_2; namely, the equation (3. 17). For $\lambda = 1$ the equation (3. 17) becomes

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where H'' and H' are spherical harmonics of order n. Hence

$$\partial U/\partial n = H_0/r^2 + H'_1/r^4 + \cdots + H'_n/r^{2n+2} + \cdots.$$

Therefore $\lim_{r\to\infty} \int_\Omega \frac{dU}{dn} dS = 4\pi H_0$. But from our assumption concerning the behavior of $U(M)$ at infinity, it follows that $\lim_{r\to\infty} \int_\Omega \frac{dU}{dn} dS = 0$. Hence $H_0 = 0$, and therefore $U(M)$ and dU/dn vanish at infinity like $1/r^2$ and $1/r^3$ respectively.

$$(1.1') \qquad k(Q,R) = -K(Q,R) - K_1(Q,R) + k_2(Q,R) + \int_{S_0} k_2(Q,x)\{K(x,R) + K_1(x,R)\} dS_x.$$

Since $K_2(Q,P)$ is continuous in Q and P, $K_2(Q,P)$ satisfies the Volterra relations

$$(1.2') \qquad \int_{S_0} k_2(Q,R)K_2(R,P)\,dS_R = \int_{S_0} k_2(R,P)K_2(Q,R)\,dS_R = k_2(Q,P) + K_2(Q,P).$$

From (1.1′) we have

$$\begin{aligned}\int_{S_0} k(Q,R)K(R,P)\,dS_R &= -\int_{S_0} K(Q,R)K(R,P)\,dS_R \\ &\quad -\int_{S_0} K_1(Q,R)K(R,P)\,dS_R + \int_{S_0} k_2(Q,R)K(R,P)\,dS_R \\ &\quad + \int_{S_0} k_2(Q,x)\,dS_x \int_{S_0} \{K(x,R)K(R,P) + K_1(x,R)K(R,P)\} dS_R \\ &= -K_1(Q,P) - K_2(Q,P) + \int_{S_0} k_2(Q,x)\,dS_x\{K(x,P) + K_1(x,P)\} \\ &\quad + \int_{S_0} k_2(Q,x)K_2(x,P)\,dS_x.\end{aligned}$$

Replacing the last integral by its value given by (1.2′), we have

$$\begin{aligned}(1.3') \quad \int_{S_0} k(Q,R)K(R,P)\,dS_R &= -K(Q,P) + k_2(Q,P) \\ &\quad + \int_{S_0} k_2(Q,x)\{K(x,P) + K_1(x,P)\} dS_x \\ &= k(Q,P) + K(Q,P)\end{aligned}$$

by (1.1′).

Since

$$-k_2(Q,R) = K_2(Q,R) + K_5(Q,R) + K_8(Q,R) + \cdots,$$

this series being uniformly convergent for Q, R on S_0; and since

$$\int_{S_0} K_i(Q,x)K_j(x,R)\,dS_x = \int_{S_0} K_j(Q,x)K_i(x,R)\,dS_x = K_{i+j+1}(Q,R)$$

it follows that

$$(1.4') \quad \int_{S_0} k_2(Q,x)K_i(x,R)\,dS_x = \int_{S_0} K_i(Q,x)k_2(x,R)\,dS_x \quad (i = 0, 1, \cdots).$$

By (1. 1′)

$$k(R,P) = -K(R,P) - K_1(R,P) + k_2(R,P)$$
$$+ \int_{S_0} k_2(R,x)\{K(x,P) + K_1(x,P)\}dS_x.$$

Making use of (1. 4′) we write this equation in the equivalent form

$$(1.5') \qquad k(R,P) = -K(R,P) - K_1(R,P) + k_2(R,P)$$
$$+ \int_{S_0} k_2(x,P)\{K(R,x) + K_1(R,x)\}dS_x.$$

Hence we have

$$\int_{S_0} k(R,P)K(Q,R)dS_R$$
$$= -K_1(Q,P) - K_2(Q,P) + \int_{S_0} k_2(x,P)K(Q,x)dS_x$$
$$+ \int_{S_0} K(Q,R)dS_R \int_{S_0} k_2(x,P)\{K(R,x) + K_1(R,x)\}dS_x$$
$$= -K_1(Q,P) - K_2(Q,P) + \int_{S_0} k_2(x,P)\{K(Q,x) + K_1(Q,x)\}dS_x$$
$$+ \int_{S_0} k_2(x,P)K_2(Q,x)dS_x.$$

By (1. 2′) this last equation may be written

$$(1.6') \qquad \int_{S_0} k(R,P)K(Q,R)dS_R$$
$$= -K_1(Q,P) + k_2(Q,P) + \int_{S_0} k_2(x,P)\{K(Q,x) + K_1(Q,x)\}dS_x$$
$$= k(Q,P) + K(Q,P)$$

from (1. 5′). Combining (1. 3′) and (1. 6′) we obtain the Volterra relations (3. 14), which was to be shown.

2.

Lemma 1a. *$K_1(Q,P)$ is bounded.*

We suppose δ to be a positive constant, and consider two cases.

Case 1. $\overline{QP} \geq 2\delta > 0$. We may write

$$4\pi^2 K_1(Q,P) = 4\pi^2 \int_{S_0 - S_0(\delta,Q) - S_0(\delta,P)} K(Q,R)K(R,P)dS_R + 4\pi^2 \int_{S_0(\delta,Q)} + 4\pi^2 \int_{S_0(\delta,P)}.$$

We denote the three terms on the right by I_1, I_2, and I_3 respectively. We now have

$$| I_1 | \leq \frac{1}{\delta^4} m(S_0)$$

$$| I_2 | \leq \frac{1}{\delta^2} \int_{S_0(\delta,Q)} \frac{| \cos \sphericalangle (QR, n_R) |}{\overline{QR}^2} dS_R$$

$$| I_3 | \leq \frac{1}{\delta^2} \int_{S_0(\delta,P)} \frac{| \cos \sphericalangle (RP, n_P) |}{\overline{RP}^2} dS_R.$$

It is obvious that $| I_1 |$ is bounded. $| I_2 |$ and $| I_3 |$ are bounded since these are surfaces of class Γ.

Case 2. $\overline{QP} < 2\delta$. We let $\delta' = 4\delta$ and write

$$4\pi^2 K_1(Q, P) = 4\pi^2 \int_{S_0 - S_0(\delta',P)} K(Q, R) K(R, P) dS_R + 4\pi^2 \int_{S_0(\delta',P)}.$$

We denote the terms on the right by J_1 and J_2 respectively. We now have

$$| J_1 | \leq \frac{1}{64\delta^4} m(S_0)$$

$$| J_2 | \leq \int_{S_0(\delta',P)} \left| \frac{\cos \sphericalangle (QR, n_R)}{\overline{QR}^2} \cdot \frac{\cos \sphericalangle (RP, n_P)}{\overline{RP}^2} \right| dS_R.$$

Since $| J_1 |$ is obviously bounded, we have only to prove that $| J_2 |$ is bounded.

Let arc PR measured along the curve of intersection of S_0 and the plane determined by n_P and R be s. Let n_P be the z-axis and the tangent plane to S_0 at P be the xy-plane. Let R have cylindrical coördinates (ρ, μ, z). In a similar manner set up a system of cylindrical coördinates at Q. Let (ρ'_1, μ', z') be the coördinates of R in this system. Denote by ρ' the projection of QR in the tangent plane at P. Let $\bar{s}$ be the arc QR measured along the curve of intersection of S_0 and the plane determined by n_Q and R. Let δ be small enough so that

$$(2.1') \quad \begin{cases} | \sphericalangle (QR, n_R) | \leq \pi; \quad | \sphericalangle (QR, n_Q) | \leq \pi, \quad | \sphericalangle n_Q, n_R) | \leq \pi \\ \rho'/\rho'_1 < 2 \\ s < 2\rho \\ ds < 2d\rho \\ \bar{s} < 2\rho'_1 \\ d\bar{s} < 2d\rho'_1. \end{cases}$$

Now we have

$$
\begin{aligned}
&|\cos \sphericalangle (RP, n_P)| = z/RP < 4\gamma\rho \text{ (see the 9th of the inequalities (2.1))}\\
&|\cos \sphericalangle (QR, n_Q)| = z'/RQ < 4\gamma\rho'_1\\
&|\sphericalangle (QR, n_R) - \sphericalangle (QR, n_Q)| \leq (n_Q, n_R)|\\
&|\cos \sphericalangle (QR, n_R) - \cos \sphericalangle (QR, n_Q)| \leq |\sphericalangle (n_Q, n_R)| < \gamma\overline{QR}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
|\cos \sphericalangle (QR, n_R)| &< |\cos \sphericalangle (QR, n_Q)| + \gamma\overline{QR}\\
&\leq 4\gamma\rho'_1 + \gamma\overline{QR}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\left| \frac{\cos \sphericalangle (RP, n_P)}{\overline{RP}^2} \right| < 4\gamma\rho/RP^2 \leq 4\gamma/\rho\\
&\left| \frac{\cos \sphericalangle (QR, n_R)}{\overline{QR}^2} \right| < 4\gamma\rho'_1/\overline{QR}^2 + \gamma/\overline{QR} \leq 5\gamma/\rho'_1 \leq 10\gamma/\rho'.
\end{aligned}
$$

We have now

$$
(2.2') \qquad |J_2| < 40\gamma^2 \int_{S_0(\delta', P)} \frac{dS_R}{\rho\rho'}.
$$

Let $C(\delta', P)$ be the circle with center P and radius δ' and lying in the tangent plane to S_0 at P. We have $|dS_R| < 2dS'$ where dS' is the projection of dS_R on the tangent plane at P. We have finally

$$
|J_2| < 80\gamma^2 \int_{C(\delta', P)} \frac{dS'}{\rho\rho'}.
$$

This last integral is known to be bounded.[30]

LEMMA 2a. *$K_1(M, N)$ is bounded uniformly as τ approaches zero.*

We suppose M, N, G on $S(\tau)$ correspond respectively to Q, P, R on S_0. For simplicity of notation we denote $S(\tau)$ by S. Since MN converges uniformly to QP as τ approaches zero, given a positive number δ, then τ_1 exists so that for all $\tau \leq \tau_1$, either $MN \geq 2\delta$ uniformly or $MN \leq 2\delta$ uniformly. Hence, we consider two cases as before.

Case 1. $MN \geq 2\delta > 0$. In this case we write

$$
4\pi^2 K_1(M, N) = 4\pi^2 \int_{S-S(\delta,M)-S(\delta,N)} K(M, G)K(G, N)\, dS_G + 4\pi^2 \int_{S(\delta,M)} + 4\pi^2 \int_{S(\delta,N)}.
$$

[30] Kellogg, *Potential Theory*, pp. 302-303.

Denoting the terms on the right by I'_1, I'_2, and I'_3 respectively, we have

$$| I'_1 | \leq \frac{1}{\delta^4} m(S)$$
$$| I'_2 | \leq \frac{1}{\delta^2} \int_{S(\delta,M)} \frac{|\cos \sphericalangle (MG, n_G)|}{\overline{MG}^2} dS_G.$$
$$| I'_3 | \leq \frac{1}{\delta^2} \int_{S(\delta,N)} \frac{|\cos \sphericalangle (GN, n_N)|}{\overline{GN}^2} dS_G.$$

Obviously $| I'_1 |$ is uniformly bounded as τ approaches zero. From Lemma 3 it follows that $| I'_2 |$ and $| I'_3 |$ are uniformly bounded.

Case 2. $MN \leq 2\delta$. Let $\delta' = 4\delta$. Now we have

$$4\pi^2 K_1(M, N) = 4\pi^2 \int_{S-S(\delta',N)} K(M, G) K(G, N) dS_G + 4\pi^2 \int_{S(\delta',N)} .$$

Denoting the terms on the right by J'_1 and J'_2 respectively, we have at once that

$$| J'_1 | \leq \frac{1}{64\delta^4} m(S),$$

and hence $| J'_1 |$ is uniformly bounded. Let $C(\delta', N)$ be the circle with center N and radius δ' and lying in the tangent plane to S at N. Let ρ_I, ρ'_I be the projections of NG and MG respectively in the tangent plane at N, and let dS'' be the projection of dS_G on the tangent plane at N. Without affecting the validity of the proof we may suppose δ small enough so that the inequalities corresponding to (2. 1′) hold uniformly as τ approaches zero. From the inequality (2. 2′) we have evidently

$$| J'_2 | < 80\gamma^2 \int_{C(\delta',N)} \frac{dS''}{\rho_I \rho'_I} .$$

But since dS''/dS', ρ_I/ρ, ρ'_I/ρ' all approach 1 uniformly as τ approaches zero, $\int_{C(\delta',N)} \frac{dS''}{\rho_I \rho'_I}$ approaches $\int_{C(\delta',P)} \frac{dS'}{\rho\rho'}$ uniformly as τ approaches zero. We have seen that $\int_{C(\delta',P)} \frac{dS'}{\rho\rho'}$ is bounded, which completes the proof of the lemma.

3.

Lemma 3a. *$g_n(M, P)$ is continuous in P for P on S_0 and M in T^+.*

Let M be a fixed point of T^+ with minimum distance D from points of S_0. It is evident that the second term on the right in equation (3. 19) is continuous

in P. For the first term we have

$$\left| \int_{S_0} k(R,P) \frac{\cos \sphericalangle (MR, n_R)}{\overline{MR}^2} dS_R - \int_{S_0} k(R,P_1) \frac{\cos \sphericalangle (MR, n_R)}{\overline{MR}^2} dS_R \right| \leq I + J;$$

where

$$I = \frac{1}{D^2} \int_{S_0(\delta,P)} | k(R,P) - k(R,P_1) | \, dS_R$$
$$J = \frac{1}{D^2} \int_{S_0 - S_0(\delta,P)} | k(R,P) - k(R,P_1) | \, dS_R.$$

We may fix δ so that I is as small as we like. Suppose $\overline{PP_1} < \delta/2$. To complete the proof we have only to show that $k(R, P_1)$ approaches $k(R, P)$ as P_1 approaches P, with $RP > \delta/2$ and $RP_1 > \delta/2$. But with this restriction on R, it is seen that each term in $k(R, P_1)$ as given by (3. 17) approaches the corresponding term in $k(R, P)$ as P_1 approaches P. Hence the lemma is true.

4.

We shall make use of the following theorem due to J. Radon: [31]

THEOREM. *Let $f_1, f_2, \cdots$ be a sequence of completely additive functions of point sets whose total variation over J is uniformly bounded and let E_0 be a closed set in J. Then we can pick out a subsequence $\{f_{n^*}\}$ of the functions f_n and a completely additive function of point sets, $f(e)$, so that for each function $\phi(P)$ continuous on E_0*

$$\lim_{n^* \to \infty} \int_{E_0} \phi(P) df_{n^*}(e_P) = \int_{E_0} \phi(P) df(e_P). \tag{4. 1'}$$

It is readily seen that the function $f(e)$ and the sequence f_{n^*} so chosen will also satisfy the relation

$$\lim_{n^* \to \infty} \int_{E_0} \phi_{n^*}(P) df_{n^*}(e_P) = \int_{E_0} \phi(P) df(e_P) \tag{4. 2'}$$

where the functions $\phi_n(P)$ are continuous and converge uniformly to $\phi(P)$. This relation is derived from the inequality

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[31] J. Radon, "Theorie und Andwendungen der absolut additiven Mengenfunktionen," *Wiener Akademie Sitzungsberichte*, 122. 2, II (a), (1913), p. 1337.

$$\left| \int_{E_0} \phi_{n^*} df_{n^*} - \int_{E_0} \phi df \right|$$
$$\leq \left| \int_{E_0} \phi_{n^*} df_{n^*} - \int_{E_0} \phi_{n^*} df \right| + \left| \int_{E_0} \phi_{n^*} df - \int_{E_0} \phi df \right|.$$

The first term on the right approaches zero as n^* becomes infinite, by the theorem above. The second term approaches zero as n^* becomes infinite on account of the uniform convergence of ϕ_{n^*} to ϕ. Hence we have the relation (4. 2′). The subsequence $f_{n^*}(e)$ satisfying (4. 1′) is said to converge to the function $f(e)$ in the *weak sense.*

Extension for normal families of surfaces. If $\{f_n(e)\}$ is a sequence of completely additive functions of point sets defined on the subsequence $\{S_n\}$ of a normal family of surfaces and whose total variations are uniformly bounded, and if the sequence $\{S_n\}$ converges to the surface S_0 of the normal family as n becomes infinite, then we may pick out a subsequence f_{n^*} of the f_n's and a completely additive function of point sets $f(e)$ defined on S_0 such that for each $\phi(P_n)$, defined for P_n on S_n and continuous in P_n and converging uniformly to $\phi(P)$ defined on S_0

$$\lim_{n^* \to \infty} \int_{S_{n^*}} \phi(P_{n^*}) df_{n^*}(e_{P_{n^*}}) = \int_{S_0} \phi(P) df(e_P).$$

This extension is immediate on account of the $1:1$ correspondence between points of S_0 and points of S_{n^*} for n^* sufficiently large. We define $f_{n^*}(e)$ on S_0 equal to $f_{n^*}(e')$, where e' on S_{n^*} is the set of points corresponding to e on S_0, and then apply (4. 2′). We say that the functions f_{n^*} on S_{n^*} converge in the *weak sense* to $f(e)$ on S_0.

5.

LEMMA 4a. *The functions $g_n(M, N)$ converge uniformly to $g_n(M, P)$ as τ approaches zero, M being a fixed point in T^+ [where N on $S(\tau)$ corresponds to P on S_0].*

For convenience, we denote $S(\tau)$ by S. Let τ be small enough so that M is interior to S as well as S_0. In the formula for $g_n(M, N)$ corresponding to (3. 19) it is obvious that the second term converges uniformly to the corresponding term in the formula (3. 19) as τ approaches zero. For the first term we have, letting G on S correspond to R on S_0,

$$(5.1') \quad \left| \int_{S_0} k(R, P) \frac{\cos \measuredangle (MR, n_R)}{\overline{MR}^2} dS_R - \int_S k(G, N) \frac{\cos \measuredangle (MG, n_G)}{\overline{MG}^2} dS_G \right| \leq I + J,$$

where

$$I = \left| \int_{S_0(\delta,P)} k(R,P) \frac{\cos \sphericalangle (MR, n_R)}{\overline{MR}^2} dS_R - \int_{S(\delta,P)} k(G,N) \frac{\cos \sphericalangle (MG, n_G)}{\overline{MG}^2} dS_G \right.$$

$$J = \left| \int_{S_0 - S_0(\delta,P)} k(R,P) \frac{\cos \sphericalangle (MR, n_R)}{\overline{MR}^2} dS_R - \int_{S - S(\delta,P)} k(G,N) \frac{\cos \sphericalangle (MG, n_G)}{\overline{MG}^2} dS_G \right. .$$

Given $\epsilon > 0$ we may fix δ so that

$$I < \epsilon/2$$

since both integrals of (5. 1′) converge. Now we let τ_1 be less than δ so that N is in $S(\delta, P)$. The integrand of the second integral of J converges uniformly to that of the first integral as τ approaches zero. Also dS_R/dS_G approaches 1 uniformly as τ approaches zero. Hence τ_1 exists so that

$$J < \epsilon/2$$

for all $\tau \leq \tau_1$. Hence the lemma is proved.

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FORMAL SOLUTIONS OF IRREGULAR LINEAR DIFFERENTIAL EQUATIONS. PART II.

By Frances Thorndike Cope.

In the first part of this paper[1] we proved the fundamental formal existence theorem for the linear differential equation

$$a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \cdots + a_n(x)y(x) = 0, \qquad a_0(x) \not\equiv 0,$$

in which the coefficients $a_i(x)$ are rational functions of x. We shall now prove the converse theorem. In § 6 we shall show the equivalence of the equation (1) to the linear differential system

$$y'_i(x) = \sum_{j=1}^{n} a_{ij}(x)y_j(x) \qquad (i = 1, 2, \cdots, n)$$

in which the $a_{ij}(x)$ are rational functions of x, and hence to the most general linear system with rational coefficients. Sections 7 and 8 are devoted to general theorems on formal reducibility and formal equivalence at ∞ respectively. The definitions and general notation of the first part will be retained.

5. Converse of fundamental theorem. The converse of Theorem I may be stated as follows:

Theorem VI. *Any set of n linearly independent formal series $y_1, y_2, \cdots, y_n$ of the general type (2) which has the properties,*

i) *that if one member of the set is of anormal form then all possible determinations of this expression are linearly dependent on $y_1, y_2, \cdots, y_n$,*

ii) *that the complete set consists of one or more subsets of the form (4), determines an essentially unique equation of the form (1), of the n-th order, for which these series are a complete set of formal solutions. It is, in fact, the equation*

$$| y_1, y'_2, \cdots, y_n^{(n-1)}, y^{(n)} | = 0. \tag{40}$$

This equation is obviously satisfied by each of the given series, since on replacing y by y_i we have two columns of the determinant identical. It is actually of n-th order, for the coefficient of $y^{(n)}$ is the Wronskian of the given

[1] This appeared in *American Journal of Mathematics*, vol. 56 (1934), pp. 411-437.

set of solutions, which is not zero since the set is linearly independent. Moreover the equation is unique, for the equations

$$a_0 y_i^{(n)} + a_1 y_i^{(n-1)} + \cdots + a_n y_i = 0 \qquad (i = 1, 2, \cdots, n)$$

constitute a system of n linear equations in the n ratios

$$a_1/a_0, a_2/a_0, \cdots, a_n/a_0,$$

in which the determinant of the coefficients is not zero, since it is precisely the Wronskian, $\Delta(y_1, \cdots, y_n)$, of the set of solutions, which are assumed linearly independent. It remains to show that the equation is actually of the type (1), that is, that the coefficients are simple formal series.

To prove this we observe first that any exponential factor which occurs in any solution will also occur in all of its derivatives, and hence in all the elements of one column of the determinant, so that it can be factored out from the equation. Thus the equation can be freed of exponential factors.

We have next to consider the possibility that it may involve $\log x$. Let $y_1, \cdots, y_m$ be a set of solutions of the form (4). Then by successive differentiations we find that

$$y_{k+1}^{(i)}(x) = A_{0i}(x) \log^k x + \cdots + \frac{k!}{j!\,(k-j)!} A_{ji}(x) \log^{k-j} x + \cdots + A_{ki}(x)$$
$$(k = 0, 1, \cdots, m-1),$$

where the coefficients

$$A_{j0}(x) = s_j(x), \qquad A_{ji}(x) = A'_{j,i-1}(x) + x^{-1} A_{j-1,i-1}(x) \qquad (0 < i),$$

are independent of k. The first m columns of the determinant then are

$$A_{00}, A_{00} \log x + A_{10}, \cdots, A_{00} \log^{m-1} x + \cdots$$
$$+ A_{j0} \frac{(m-1)\cdots(m-j)}{j!} \log^{m-j-1} x + \cdots + A_{m-1,0},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$A_{0n}, A_{0n} \log x + A_{1n}, \cdots, A_{0n} \log^{m-1} x + \cdots$$
$$+ A_{jn} \frac{(m-1)\cdots(m-j)}{j!} \log^{m-j-1} x + \cdots + A_{m-1,n},$$

and by subtracting $\log^{k-1} x$ times the first column from the k-th column $(2 \leq k \leq m)$ the leading term of each element is eliminated from all columns 2 to m, i. e. the highest power of $\log x$ is reduced by one. Then by subtracting $(k-1) \log^{k-2} x$ times the second column from the k-th column $(3 \leq k \leq m)$ the highest power of $\log x$ which appears is again reduced, and by continuing

the process we can eliminate step by step all the terms involving $\log x$ in these columns of the determinant. Since this set of solutions is typical of those in which $\log x$ occurs, the expansion of the determinant is evidently free from $\log x$. Thus the equation (40), since it does not involve $\log x$ or any exponential factor, is of the required form, that is, has coefficients which are formal series in descending powers of $x^{1/mp}$.

Since any determination of a series $y_i(x)$ of anormal form is linearly dependent on $y_1, y_2, \cdots, y_n$, it must also be a solution of the equation (40). Consequently the equation obtained is essentially independent of the choice of $\arg x^{1/mp}$, $\arg x^{1/p}$ being assumed fixed. Its coefficients therefore do not involve fractional powers of $x^{1/p}$ but are expressible as formal series in $x^{1/p}$.

6. Reduction of general linear system to a single equation. It is well known that any system of linear differential equations can be reduced to a system of linear equations of the first order, homogeneous if the original system is homogeneous. For example, the homogeneous equation of the n-th order in one variable, equation (1), can always be reduced to a system of n linear homogeneous equations of the first order in n variables, in particular by taking

$$y_1(x) = y(x), \qquad y_i(x) = y'_{i-1}(x) \qquad (i = 2, 3, \cdots, n),$$

in which case the linear system is

$$\begin{aligned} y'_i(x) &= y_{i+1}(x) \\ a_0(x)y'_n(x) &= -a_1(x)y_n(x) - \cdots - a_n(x)y_1(x) \end{aligned} \qquad (i = 1, 2, \cdots, n-1)$$

and has rational coefficients if the coefficients of the equation (1) are rational. It is also true, though less familiar, that any linear system

$$y'_i(x) = \sum_{j=1}^{n} a_{ij}(x)y_j(x) \qquad (i = 1, 2, \cdots, n) \tag{41}$$

with rational coefficients can be reduced to a single equation (1) of the n-th order with rational coefficients.

To prove this we let

$$y(x) = \lambda_1(x)y_1(x) + \lambda_2(x)y_2(x) + \cdots + \lambda_n(x)y_n(x)$$

where the functions $\lambda_i(x)$ are for the present arbitrary rational functions. Then on differentiating repeatedly and substituting each time the values of $y'_i(x)$ from the equations (41) we have $n+1$ equations

$$y^{(i-1)}(x) = \sum_{j=1}^{n} \lambda_{ij}(x) y_j(x) \qquad (i = 1, 2, \cdots, n+1), \tag{42}$$

in which the right-hand terms are linear combinations of $y_1(x), \cdots, y_n(x)$. Consequently there must be at least one linear homogeneous equation between the left-hand terms, that is, an equation of the form

$$a_0(x) y^{(n)}(x) + \cdots + a_n(x) y(x) = 0$$

in which at least one of the coefficients is different from zero.

In fact we can choose the $\lambda_i(x)$ so that $a_0(x) \not\equiv 0$, since they can be so chosen that the determinant of the coefficients of $y_1, y_2, \cdots, y_n$ in the first n of the above $n+1$ equations is not identically zero. The elements of the i-th row of this determinant are

$$\gamma_{ij} = \lambda_j^{(i-1)} + P_{ij} \qquad (j = 1, 2, \cdots, n),$$

where P_{ij} is a polynomial in $\lambda_1, \cdots, \lambda_n$, their first $i-2$ derivatives, a_{rs} $(r, s = 1, \cdots, n)$, and their derivatives. Hence at any point $x = x_0$ which is not a pole of one of the coefficients $a_{rs}(x)$, the elements of the determinant can be given arbitrary values by assigning suitable values first to $\lambda_1(x_0), \cdots, \lambda_n(x_0)$, then to $\lambda'_1(x_0), \cdots, \lambda'_n(x_0)$, and to the derivatives of higher orders successively, up to $\lambda_1^{(n-1)}(x_0), \cdots, \lambda_n^{(n-1)}(x_0)$. It is always possible to determine a rational function $\lambda_j(x)$ such that $\lambda_j^{(i)}(x_0) = c_{ij}$, where the c_{ij} $(i = 0, 1, \cdots, n)$ are arbitrary constants. In fact, the polynomial

$$\lambda_j(x) \equiv c_{0j} + c_{1j}(x - x_0) + \cdots + \frac{c_{ij}}{i!}(x - x_0)^i + \cdots + \frac{c_{n-1,j}}{(n-1)!}(x - x_0)^{n-1}$$

is such a function. Thus it is possible to determine $\lambda_1(x), \cdots, \lambda_n(x)$ so that $|\gamma_{ij}(x_0)| \neq 0$, and hence $a_0(x) \not\equiv 0$, and we have an equation of the n-th order in $y(x)$. Moreover the coefficients in this equation are rational combinations of the coefficients of the linear system and therefore are rational functions of x.

The solution of the linear system in terms of $y(x)$ is then obtained by solving simultaneously the first n equations of the set (42) for $y_1, \cdots, y_n$. Since $|\gamma_{ij}|$, the determinant of the coefficients, has been made different from zero, Cramer's rule may be applied, and the solution will be of the form

$$y_i(x) = \beta_{i1} y^{(n-1)}(x) + \cdots + \beta_{in} y(x) \qquad (i = 1, 2, \cdots, n) \tag{43}$$

where the $\beta_{i1}, \cdots, \beta_{in}$ are rational functions of x. Thus each solution of the n-th order equation in $y(x)$ will lead to a solution of the linear system, and we have

THEOREM VII. *Any system of homogeneous linear differential equations of the first order with rational coefficients can be reduced to a single linear homogeneous differential equation in one variable with rational coefficients.*

We may therefore regard either the single homogeneous equation of n-th order in one variable or the system of n homogeneous equations of the first order in n variables as essentially equivalent to the most general homogeneous linear system. Since, however, the system of n equations of the first order is often the more convenient to use, we shall derive the theorem analogous to Theorem I for such a system.

From the equations (43) which give the solutions of the linear system (41) in terms of the solutions of a single equation (1), we can show that to each subset,

$$y^{[m+1]}(x) = s_0(x)\log^m x + \cdots + \frac{m!}{j!(m-j)!} s_j(x)\log^{m-j} x + \cdots + s_m(x) \qquad (m = 0, 1, \cdots, k),$$

of solutions of the single equation there corresponds a similar subset,

$$y_i^{[m+1]}(x) = \alpha_{i0}(x)\log^m x + \cdots + \frac{m!}{j!(m-j)!} \alpha_{ij}(x)\log^{m-j} x + \cdots + \alpha_{im}(x) \qquad (i = 1, 2, \cdots, n;\ m = 0, 1, \cdots, k)$$

of solutions of the corresponding system.

If $y^{[m+1]}(x)$ is of the form (2) we find that its i-th derivative is

$$y^{[m+1](i)}(x) = A_{0i}(x)\log^m x + \cdots + \frac{m!}{j!(m-j)!} A_{ji}(x)\log^{m-j} x + \cdots + A_{mi}(x),$$

where $A_{ji}(x)$ is defined, for $(j = 0, 1, \cdots, m)$, by the equations

$$A_{j0}(x) = s_j(x), \quad A_{ji}(x) = A'_{j,i-1}(x) + x^{-1}A_{j-1,i-1}(x) \qquad (i = 1, 2, \cdots, n).$$

Consequently, after substituting these values in the equations (43), we can collect coefficients of powers of $\log x$ and have, for $(i = 1, 2, \cdots, n)$ and $(m = 0, 1, \cdots, k)$,

$$y_i^{[m+1]}(x) = \sum_{j=0}^{m} \frac{m!}{j!(m-j)!} [\beta_{i1}(x)A_{j,n-1}(x) + \cdots + \beta_{in}(x)A_{j0}(x)]\log^{m-j} x,$$

which we may write in the form

$$y_i^{[m+1]}(x) = \sum_{j=0}^{m} \frac{m!}{j!(m-j)!} \alpha_{ij}(x)\log^{m-j} x \quad (i = 1, 2, \cdots, n;\ m = 0, 1, \cdots, k),$$

where the coefficients $\alpha_{ij}(x)$ do not depend on m, that is, in a form similar to that of $y^{[m+1]}(x)$. Hence each such set of solutions, $y^{[m+1]}(x)$ $(m=0,1,\cdots,k)$, leads to a corresponding set of solutions of similar form, $y_1^{[m+1]}(x),\cdots,y_n^{[m+1]}(x)$ $(m=0,1,\cdots,k)$.

From the equation by which $y(x)$ is introduced it is clear that if a set of solutions of the linear system (41) is linearly dependent then so is the corresponding set of solutions of the equation (1), for if there exist constants $c_1,\cdots,c_n$, not all zero, such that

$$c_1y_i^{[1]}+c_2y_i^{[2]}+\cdots+c_ny_i^{[n]}=0 \qquad (i=1,2,\cdots,n),$$

then

$$c_1y^{[1]}+\cdots+c_ny^{[n]}=\sum_{i=1}^{n}\lambda_i[c_1y_i^{[1]}+\cdots+c_ny_i^{[n]}]=0$$

also. Therefore as a corollary of Theorem I we have

THEOREM VIII. *Any linear system* (41) *in which the coefficients are formal series in descending powers of $x^{1/p}$, p being a positive integer, has always a complete set of n distinct formal solutions of the general type*

$$y_i(x)=s_{i0}(x)\log^k x+s_{i1}(x)\log^{k-1}x+\cdots+s_{ik}(x) \quad (i=1,2,\cdots,n),$$

and the complete set consists of subsets of the form

$$y_i^{[j+1]}(x)=\sum_{l=0}^{j}\frac{j!}{l!(j-l)!}s_{il}(x)\log^{j-l}x \quad (i=1,2,\cdots,n;\ j=0,1,\cdots,k).$$

From the complete set of solutions for the homogeneous equation (1) a particular solution of the corresponding non-homogeneous equation

$$a_0(x)y^{(n)}(x)+\cdots+a_n(x)y(x)=r(x), \tag{44}$$

where $r(x)$ is a formal series in descending powers of $x^{1/p}$, can be found by the method of variation of parameters. This solution is

$$Y(x)=c_1(x)y_1(x)+\cdots+c_n(x)y_n(x),$$

where

$$c'_i(x)=r(x)\Delta_{ni}/\Delta\,^2$$

and Δ_{ni} denotes the cofactor of the element in the n-th row and i-th column of $\Delta(y_1,y_2,\cdots,y_n)$.

Let the exponential factors of $y_1,y_2,\cdots,y_n$ be denoted by

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[2] By the process used to show that equation (40) involves neither $\log x$ nor exponential factors, Δ can be shown to be of the form (7).

$$e^{Q_1(x)}, e^{Q_2(x)}, \cdots, e^{Q_n(x)}$$

respectively. Then the quotient Δ_{ni}/Δ has the exponential factor $e^{-Q_i(x)}$. From the proof of the lemma on page 26 it is clear that the solution of the non-homogeneous equation of first order has the same exponential factor as the right-hand term of the equation. Hence $c_i(x)$ has the exponential factor $e^{-Q_i(x)}$, and the products $c_i(x)y_i(x)$ $(i = 1, 2, \cdots, n)$ have no exponential factors. The particular solution $Y(x)$ of the non-homogeneous equation of the n-th order is therefore an *elementary* formal solution and in fact one in which no exponential factor occurs, and we have the

THEOREM IX. *Any non-homogeneous linear differential equation* (44) *in which both the coefficients and the right-hand terms are formal series in descending powers of* $x^{1/p}$, *has at least one particular formal solution which is of the elementary type* (2) *and in fact does not involve an exponential factor.*

The same general method may be employed to determine a particular solution of the non-homogeneous linear system

$$y'_i(x) = \sum_{j=1}^{n} a_{ij}(x)y_j(x) + r_i(x) \qquad (i = 1, 2, \cdots, n)$$

in which the $r_i(x)$ as well as the coefficients $a_{ij}(x)$ are formal series, and an elementary solution containing no exponential factor is obtained in this case also.

7. Reducibility. In the course of the proof of Theorem I we have seen that an equation of the type (1′) and of order greater than one is always reducible if the basic integer p is suitably chosen. For an arbitrarily chosen admissible basic integer, however, the equation is not necessarily reducible. The reducibility of the equation is intimately related to the character of its complete set of formal solutions. In order to show the relationship we introduce the concept of a *natural family* of solutions.

Let $y_1(x), \cdots, y_n(x)$ be a set of n linearly independent solutions of the equation (1′). Then any linear combination of them with constant coefficients is also a solution of this equation. The aggregate of such combinations which are of the form (2) constitutes a family of solutions, which has the property that if one determination of multiple-valued solution belongs to it then so do the other determinations. Any set of elementary solutions of the equation which has these two properties, namely,

i) *that any member of the set can be expressed as a linear combination of m linearly independent members of the set,*

ii) *that if one determination of a multiple-valued solution belongs to the set then so do its other determinations*

is called a *natural family* of solutions, and the number m of linearly independent members is called the *order* of the family. For example, a single non-logarithmic solution of normal form constitutes a natural family. This definition of natural family clearly, like those of reducibility and of normal and anormal series, is relative to a particular basic integer p.

It is not difficult to show that among the various sets of m linearly independent members, by which a given natural family can be generated, there is at least one of the type described in Theorem I and required by the hypotheses of Theorem VI. Hence we have the

THEOREM VI′. *Any natural family of solutions determines an equation of type* (1′), *of order equal to the order of this family, which has the members of this natural family, and only these, as its formal solutions.*

If we add to any given natural family, say F_1, of order n_1, one or more new members such that the set thus formed is also a natural family, say F_2, of order n_2 $(> n_1)$, the new family F_2 will determine an equation of order n_2. If the two corresponding equations are denoted by $L_1(y) = \theta$ and $M(y) = 0$ respectively, then it is clear, since any solution of the first is also a solution of the second, that the second may be expressed as $L_2(L_1(y)) = 0$, that is, that L_1 is a symbolic factor of M. Similarly, to any expanding sequence of natural families $F_1, F_2, \cdots, F_m$, there corresponds a sequence of equations,

$$L_1(y) = 0,\ L_2(L_1(y)) = 0, \cdots, L_m(L_{m-1} \cdots (L_1(y))) = 0,$$

such that the formal solutions of the i-th equation are precisely the members of F_i $(i = 1, 2, \cdots, m)$.

On the other hand, if the differential expression $L(y)$ can be factored symbolically, that is, if $L(y) = M(N(y))$, where M and N are differential operators of the same type as L, then the solutions of the equation $N(y) = 0$ are solutions of the equation $L(y) = 0$ also, and hence form a sub-family of the natural family of solutions of the latter equation. Consequently any factorization $L \equiv L_m L_{m-1} \cdots L_1$ of the differential operator $L(y)$ determines an expanding sequence of natural families $F_1, F_2, \cdots, F_m$ such that F_i consists of the formal solutions of the equation

$$L_i(L_{i-1} \cdots (L_1(y))) = 0 \qquad (i = 1, 2, \cdots, m).$$

Furthermore, if the factors $L_1, \cdots, L_m$ are irreducible, then the corresponding sequence of natural families is such that there is no intermediate natural family distinct from those of the sequence, and conversely.

This result is precisely analogous, in statement and proof, to the theorem on reducibility stated by Birkhoff for the linear difference equations.[3] It may be stated as follows:

THEOREM X. *To any decomposition of $L(y)$, of order n, into irreducible symbolic factors $L_1, L_2, \cdots, L_m$ such that $L \equiv L_m L_{m-1} \cdots L_1$, there corresponds a sequence of natural families $F_1, F_2, \cdots, F_m$, each containing the preceding as a sub-family, such that there exist no intermediate natural families, and such that the general solution of $L_1(y) = 0$ is furnished by F_1, of $L_1 L_2(y) = 0$ by F_2, etc.*

Conversely, to any set of natural families $F_1, F_2, \cdots, F_m$ (of formal solutions of $L(y) = 0$), each containing the preceding as a sub-family, but such that there exist no intermediate natural families, there corresponds an irreducible factorization of $L(y)$, which is essentially unique.

8. Equivalence. For certain purposes it is convenient to focus attention on the set of *distinct* natural families into which a given natural family can be divided, rather than on an expanding sequence of natural families contained in it. It is clear that each set of solutions (4), with their various determinations if they are multiple-valued, determines a distinct natural family of solutions of the equation (1). Similarly the complete set of solutions of the linear system

$$y'_i(x) = \sum_{j=0}^{n} a_{ij}(x) y_j(x) \qquad (i = 1, 2, \cdots, n) \tag{45}$$

consists of distinct sets of the form

$$y^{[\lambda]}_{i1}(x), y^{[\lambda]}_{i2}(x), \cdots, y^{[\lambda]}_{i,k+1}(x) \quad (\lambda = 1, 2, \cdots, m; i = 1, 2, \cdots, n), \tag{46}$$

where $y^{[1]}_{ij}(x), \cdots; y^{[m]}_{ij}$ denote the m distinct determinations of the solution $y_{ij}(x) = s_{i0}(x) \log^{j-1} x + \cdots + s_{i,j+1}(x)$ for $y_i(x)$.

This decomposition of the fundamental set of solutions into distinct sets which correspond to distinct natural families is useful in studying the equivalence of linear systems. In matrix notation the system (45) is written as $Y'(x) = A(x) Y(x)$ where $A(x)$ denotes the matrix $\| a_{ij}(x) \|$, and the matrix solution $S(x)$ is the n-rowed square matrix formed by the fundamental set

[3] *Loc. cit.*, pp. 238-241.

of solutions. (Hence $|S(x)| \neq 0$. A linear transformation $Y(x) = B(x)\bar{Y}(x)$ $(|B(x)| \neq 0)$, in which the elements of $B(x)$ are series of the form $b(x) = b_0 + b_1 x^{-1/mp} + b_2 x^{-2/mp} + \cdots$ then takes the original equation (45) into another

$$\bar{Y}'(x) = \bar{A}(x)\bar{Y}(x) \qquad (\bar{A}(x) = B^{-1}(x)[A(x)B(x) - B'(x)]),$$

of the same type, and the two equations are said to be *formally equivalent at* ∞. If all the elements of $B(x)$ are of normal form the equations are called *properly equivalent*; otherwise they are called *improperly equivalent.*[4] If $S(x) = B(x)E(x)$ $(|B(x)| \neq 0)$, so that $E(x)$ is a solution of the new equation, then we have $\bar{A}(x) = E'(x)E^{-1}(x)$.

From the form of the sets (46) we find that such a factorization of $S(x)$ is always possible. Let the first $m(k+1)$ columns of $S(x)$ consist of a set of solutions (46), with the common exponential factor $e^{Q(x)}$; let the highest positive power of x which occurs in any of the coefficients $s_{ij}(x)$ be the r-th;[5] let $e_{1\lambda}$ denote the λ-*th* determination of $x^r e^{Q(x)}$; and let $s_{ij}^{[\lambda]}(x)$ be the λ-th determination of $s_{ij}(x)/e_{1\lambda}$. Then $s_{ij}^{[\lambda]}(x)$ is a simple formal series with no terms in positive powers of $x^{1/mp}$, and these columns of $S(x)$ can evidently be obtained in a product $B(x)E(x)$ if the first $m(k+1)$ columns of $B(x)$ are

$$\underbrace{\underbrace{\begin{matrix} s_{10}^{[1]}, \cdots, s_{1k}^{[1]}; \\ \cdot \quad \cdot \quad \cdot \\ s_{n0}^{[1]}, \cdots, s_{nk}^{[1]}; \end{matrix}}_{k+1 \text{ columns}} \begin{matrix} \cdots; \\ \\ \cdots; \end{matrix} \underbrace{\begin{matrix} s_{10}^{[m]}, \cdots, s_{1k}^{[m]} \\ \cdot \quad \cdot \quad \cdot \\ s_{n0}^{[m]}, \cdots, s_{nk}^{[m]} \end{matrix}}}_{m \text{ sets of } k+1 \text{ columns each}}$$

and the first $m(k+1)$ rows of $E(x)$ are

$$\begin{matrix}
e_{11}, e_{11}\log x, \cdots, & e_{11}\log^k x; & 0, \cdots\cdots\cdots\cdots\cdots\cdots, 0 \\
0, \; e_{11}, \qquad \cdots, & k e_{11}\log^{k-1} x; & 0, \cdots\cdots\cdots\cdots\cdots\cdots, 0 \\
\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot & & \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
0, \cdots\cdots, & e_{11} & ; 0, \cdots\cdots\cdots\cdots\cdots\cdots \\
\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot & & \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
0, \cdots\cdots\cdots\cdots\cdots\cdots & , 0; & e_{1m}, \cdots, e_{1m}\log^k x; \; 0, \cdots, 0 \\
\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot & & \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
0, \cdots\cdots\cdots\cdots\cdots\cdots & , 0; & 0, \quad \cdots, e_{1m}; \qquad 0, \cdots, 0.
\end{matrix}$$

Similarly each set of solutions (46), i. e., each distinct natural family of

[4] Cf. Birkhoff, *loc. cit.*, p. 242.

[5] If no positive powers occur let $r = 0$.

solutions, determines a set of columns of $B(x)$ and of rows of $E(x)$, and we have

THEOREM XI. *An arbitrary linear system*

$$Y'(x) = A(x)Y(x), \tag{45'}$$

in which the elements of the matrix $A(x)$ are formal series in descending powers of $x^{1/p}$, is equivalent (improperly) to a normal system

$$\bar{Y}'(x) = E'(x)E^{-1}(x)\bar{Y}(x),$$

in which the matrix $E(x)$ consists of zeros except for blocks, along the principal diagonal, of the form

$$\begin{array}{llll} e_{\nu\lambda}, & e_{\nu\lambda}\log x, \cdots, & & e_{\nu\lambda}\log^{k_\nu} x \\ 0, & e_{\nu\lambda}, & \cdots, & k_\nu e_{\nu\lambda}\log^{k_\nu-1} x \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot \\ 0, & \cdots\cdots\cdots, & & e_{\nu\lambda}, \end{array}$$

where $e_{\nu\lambda}$ denotes the λ-th determination of the factor $x^r e^{Q_\nu(x)}$ in the ν-th set of solutions (46) of the system (45′).

To illustrate the application of this result consider the special case in which the formal solutions are all non-logarithmic and of normal type. Then there are n distinct natural families, the matrix $E(x)$ consists entirely of zeros except for the elements $x^r e^{Q_1(x)}, \cdots, x^r e^{Q_n(x)}$ on the principal diagonal, and the normal form is

$$y'_i(x) = [Q'_i(x) + rx^{-1}]y_i(x) \qquad (i = 1, 2, \cdots, n).$$

RADCLIFFE COLLEGE.

ON THE ALGEBRAIC PROBLEM CONCERNING THE NORMAL FORMS OF LINEAR DYNAMICAL SYSTEMS.

By JOHN WILLIAMSON.

Introduction. Let m be the number of degrees of freedom of a linear conservative dynamical system and let the point $(q_1, q_2, \cdots, q_m, p_1, p_2, \cdots, p_m)$ of the phase space be denoted by $x = (x_1, x_2, \cdots, x_{2m})$. A system of $2m$ ordinary differential equations of the first order, which are homogeneous, linear and do not contain t explicitly, is a canonical system if, and only if, there exists a symmetric matrix A, such that the differential equations may be written in the form

$$\text{(i)} \qquad Bdx/dt = Ax,$$

where B is the skew symmetric matrix $\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$, and E the unit matrix of order m. In fact, apart from a factor 2, A is simply the matrix of the $2m$-ary quadratic form, which represents the Hamiltonian function. A non-singular matrix P is said to be a *Hamiltonian* matrix, if the transformation $x = Py$ sends every differential system of the form (i) into a differential system of the same form.

It has been pointed out by Wintner [1] that, if the system (i) is transformed into the system

$$\text{(ii)} \qquad Bdy/dt = Cy$$

by the transformation $x = Py$, then P is a Hamiltonian matrix if and only if

$$\text{(iii)} \qquad P'BP = sB \quad \text{and} \quad P'AP = sC.$$

In the following pages we use this result to determine a normal form for the system of equations (i). Equations (iii) imply,

$$P'(A - \lambda B)P = s(C - \lambda B),$$

or

$$P'_1(A - \lambda B)P_1 = \pm\, (C - \lambda B),$$

where $P_1 = (1/\sqrt{|\,s\,|})P$. Since $\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} B \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} = -B$, we have either

[1] A. Wintner, "On the linear conservative dynamical systems," *Annali di matematica pura ed applicata*, ser. 4, tomo 13 (1934-1935).

$$P'_1(A-\lambda B)P_1 = C-\lambda B \quad \text{or} \quad P'_2(A-\lambda B)P_2 = C_2-\lambda B,$$

where $P_2 = P_1\begin{pmatrix}0 & E\\ E & 0\end{pmatrix}$ and $C_2 = \begin{pmatrix}0 & E\\ E & 0\end{pmatrix} C \begin{pmatrix}0 & E\\ E & 0\end{pmatrix}$. In fact, if C is written as a matrix of matrices of order m, so that $C = \begin{pmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{pmatrix}$, then $C_2 = \begin{pmatrix}C_{22} & C_{21}\\ C_{12} & C_{11}\end{pmatrix}$. It is therefore apparent that a normal form for the system (i) can be obtained from a suitable canonical form of the matrix pencil $A-\lambda B$.[2]

Accordingly we first consider the purely algebraic problem of determining a canonical form for a pencil $A-\lambda B$ where A is symmetric and B is skew symmetric and non-singular. For the sake of generality we assume that the elements of the matrices A and B lie in a commutative field K and that the transformation matrices are restricted to have elements in the same field. Later we particularize B to be the matrix $\begin{pmatrix}0 & -E\\ E & 0\end{pmatrix}$ and so obtain results applicable to the original dynamical problem. In the final section the simplifications, which arise when K is the field of all real numbers, are considered and a list of the possible normal forms of the matrix A in (i) is given for the case of two degrees of freedom ($m=2$). These normal forms depend on the elementary divisors of $A-\lambda B$ which may be *real, complex or pure imaginary.* It is interesting to note that, if any of the elementary divisors of $A-\lambda B$ are pure imaginary, the elementary divisors alone are not sufficient to determine the normal form.

I. Let A and B be two square matrices of order n with elements in a commutative field K of characteristic zero. Further, let A be symmetric and B be skew-symmetric and non-singular, so that $A = A'$, $B = -B'$ and $|B| \neq 0$. If M is any matrix with elements in K, which is similar to AB^{-1}, there exists a non-singular matrix P, with elements in K, such that $P^{-1}AB^{-1}P = M$. Hence, if λ is any indeterminate, $P^{-1}(A-\lambda B)B^{-1}P = M-\lambda E$. Accordingly,

$$P'(B^{-1})'(A-\lambda B)B^{-1}P = P'(B^{-1})'P(M-\lambda E) = R(M-\lambda E),$$

where $R = P'(B^{-1})'P$. If $C = B^{-1}P$, we may write this last equation in the form

$$C'(A-\lambda B)C = R(M-\lambda E). \tag{1}$$

As a consequence of (1),

$$R = C'BC \quad \text{and} \quad RM = C'AC,$$

[2] Cf. C. Lanczos, "Eine neue Transformationstheorie linearer kanonischer gleichungen," *Annalen der Physik*, 5 Folge; ser. 653, Band 20 (1934).

so that R is skew symmetric and RM is symmetric. Therefore

$$(2) \qquad RM = (RM)' = M'R' = -M'R.$$

The pencil $A - \lambda B$ is *equivalent* under a non-singular *congruent transformation* with elements in K to the pencil $RM - \lambda R$ and we may, without any risk of confusion, simply say that the two pencils are equivalent. We shall proceed to determine a canonical form for the pencil $A - \lambda B$ by choosing a suitable form for the matrix M and by reducing the matrix R. We first notice that, if W is a non-singular matrix with elements in K and if

$$(3) \qquad W'R(M - \lambda E)W = S(M - \lambda E),$$

then

$$(4) \qquad S = W'RW$$

and

$$W'RMW = SM = W'RWM \text{ by } (4),$$

so that, since $W'R$ is non-singular,

$$(5) \qquad MW = WM.$$

Hence, in the reduction of the matrix R, we are only at liberty to use transformations, whose matrices are commutative with M.

Further, if Q is a non-singular matrix satisfying the equation

$$(6) \qquad QM = -M'Q,$$

it follows easily from (2) that

$$(7) \qquad R = QG,$$

where

$$(8) \qquad GM = MG.$$

If M is a *diagonal block* matrix,

$$(9) \qquad M = [M_1, M_2, \cdots, M_t],$$

where M_i is a square matrix of order e_i, we may write G as a matrix of matrices,

$$G = (G_{ij}) \qquad (i, j = 1, 2, \cdots, t),$$

where G_{ij} is a matrix of e_i rows and e_j columns.

If Q_i is a non-singular matrix of order e_i, such that

$$Q_i M_i = -M'_i Q_i \qquad (i = 1, 2, \cdots, t),$$

then the diagonal block matrix

$$Q = [Q_1, Q_2, \cdots, Q_t] \tag{10}$$

is non-singular and satisfies (6).

We now prove,

LEMMA 1. *If $Q'_i = \rho_i Q_i$ $(i = 1, 2, \cdots, t)$, where $\rho_i = \pm 1$, and G_{11} is non-singular, there exists a non-singular matrix W commutative with M, such that $W'QGW = QH$, where $H = (H_{ij})$ $(i, j = 1, 2, \cdots, t)$, and $H_{11} = G_{11}$, $H_{1j} = H_{j1} = 0$ $(j \neq 1)$.*

Proof. Let

$$W = \begin{pmatrix} E_1 & -G_{11}^{-1}G_{12} & -G_{11}^{-1}G_{13} & \cdots & -G_{11}^{-1}G_{1t} \\ 0 & E_2 & 0 & \cdots & 0 \\ 0 & 0 & E_3 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & E_t \end{pmatrix},$$

where E_i is the unit matrix of order e_i.

Since the matrix G satisfies (8), $G_{ij}M_j = M_i G_{ij}$ and consequently, $-G_{11}^{-1}G_{1j}M_j = -G_{11}^{-1}M_1 G_{1j} = -M_1 G_{11}^{-1}G_{1j}$. Hence the matrix W is commutative with M.

Since R is skew symmetric, it follows from (7) that

$$Q_i G_{ij} = -G'_{ji}Q'_j = -\rho_j G'_{ji} Q_j. \tag{11}$$

The element in the j-th row, $j \neq 1$, of the first column of $W'Q$ is

$$-G'_{1j}(G_{11}^{-1})'Q_1.$$

But

$$-G'_{1j}(G_{11}^{-1})'Q_1 = -\rho_1 G'_{1j}Q_1 G_{11}^{-1} = -\rho_1^2 Q_j G_{j1} G_{11}^{-1} = -Q_j G_{j1} G_{11}^{-1}$$

by (11) and the definition of ρ_1. Therefore $W'Q = QL$, where L is the matrix obtained from W' by substituting $-G_{j1}G_{11}^{-1}$ for $-G'_{1j}(G_{11}^{-1})'$ as the element in the j-th place of the first column, $j \neq 1$. Since the element in the first row and the first column of the product LGW is G_{11}, while all other elements in the first row or column are zero, the lemma is proved.

If the diagonal block matrix M in (9) is such that every matrix G commutative with M is also a diagonal block matrix $G = [G_1, G_2, \cdots, G_t]$, where G_i is a square matrix of order e_i, then

$$G_i M_i = M_i G_i \qquad (i = 1, 2, \cdots, t). \tag{12}$$

Further, as a consequence of (7), R is a diagonal block matrix $[R_1, R_2, \cdots, R_t]$, where

$$R_i = Q_i G_i \qquad (i = 1, 2, \cdots, t), \tag{13}$$

and since W is commutative with M, $W = [W_1, W_2, \cdots, W_t]$ and the matrix S defined by (4) is a diagonal block matrix $[S_1, S_2, \cdots, S_t]$, where

$$S_i = W'_i R_i W_i \qquad (i = 1, 2, \cdots, t). \tag{14}$$

But, apart from the suffixes i, equations (12), (13) and (14) are the same as (8), (7) and (4) respectively. *Therefore, in reducing R to S, we need only consider the reduction of each block R_i separately by matrices commutative with M_i.*

2. *Form of the matrix M.* Since the elements of the matrices A and B lie in the field K, the invariant factors of the pencil $A - \lambda B$ are polynomials in λ with coefficients in K. We shall call the powers of the distinct irreducible factors of the invariant factors, *the elementary factors* (with respect to K) of the pencil. Since A is symmetric and B is skew symmetric, the invariant factors are unaltered, except perhaps in sign, by the interchange of λ and $-\lambda$. Hence each invariant factor is the product of an even polynomial in λ by a power of λ. Accordingly the elementary factors of the pencil $A - \lambda B$ are of three types:

Type a. $[p(\lambda)]^k$ together with $[p(-\lambda)]^k$, where $p(\lambda)$ is an irreducible polynomial but is not an even polynomial in λ and $p(\lambda) \neq \lambda$.

Type b. $[h(\lambda)]^k$ where $h(\lambda) = p(\lambda^2)$ is an even irreducible polynomial in λ.

Type c. λ^k.

We now proceed to determine matrices with elementary factors of types (a), (b) and (c) respectively.[3]

Type (a). Let $p(\lambda)$ be of degree m and let p be any matrix of order m with elements in K, whose characteristic equation is $p(\lambda)$[4] $= 0$, and let e be the unit matrix of order m. Then, if

$$\pi = \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix}, \quad \phi = \begin{pmatrix} e & 0 \\ 0 & -e \end{pmatrix}, \quad \epsilon = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}, \tag{15}$$

[3] If D is a matrix with elements in K we shall mean by the elementary factors of D, the elementary factors of the pencil $D - \lambda E$, where E is the unit matrix.

[4] We may take as the matrix p the companion matrix of $p(\lambda)$. Cf. J. Williamson, "The equivalence of non-singular pencils of hermitian matrices in an arbitrary field," *American Journal of Mathematics*, vol. 57 (1935), p. 475.

the matrix

(16)
$$N = \begin{bmatrix} \pi & \phi & 0 & \cdot & 0 \\ 0 & \pi & \phi & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \phi \\ 0 & 0 & 0 & \cdot & \pi \end{bmatrix}$$

of order k, considered as a matrix of matrices of order $2m$, has the elementary factors $[p(\lambda)]^k$, $[p(-\lambda)]^k$. For $[p(N)]^k[p(-N)]^k = 0$ and N satisfies no equation of lower degree. We now write (16) in the more convenient form

(17)
$$N = \pi E + \phi U,$$

where

$$E = \begin{bmatrix} \epsilon & 0 & \cdot & 0 \\ 0 & \epsilon & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \epsilon \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 0 & \epsilon & 0 & \cdot & 0 \\ 0 & 0 & \epsilon & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \epsilon \\ 0 & 0 & 0 & \cdot & 0 \end{bmatrix},$$

and proceed to determine a non-singular matrix V satisfying

(18)
$$VN = -N'V.$$

If

(19)
$$T = \begin{bmatrix} 0 & 0 & \cdot & 0 & \epsilon \\ 0 & 0 & \cdot & \epsilon & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \epsilon & \cdot & 0 & 0 \\ \epsilon & 0 & \cdot & 0 & 0 \end{bmatrix},$$

we see immediately that $T^2 = E$ and that

(20)
$$TU = U'T.$$

Further, we can determine a non-singular symmetric matrix q such that [5]

(21)
$$qp = p'q.$$

Since the matrix τ, defined by

(22)
$$\tau = \begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix},$$

satisfies the equation

(23)
$$\tau\phi = -\phi\tau = -\phi'\tau,$$

it follows that the matrix

(24)
$$V = q\tau T$$

[5] J. Williamson, *loc. cit.*, p. 490.

is a non-singular skew symmetric matrix satisfying (18). In fact,

$$\begin{aligned} VN &= q\tau T(\pi E + \phi U) \text{ by } (17) \\ &= q\tau(\pi E + \phi U')T \text{ by } (20) \\ &= -(\pi' E + \phi' U')q\tau T \text{ by } (23) \\ &= -N'V. \end{aligned}$$

Type b. The characteristic equation of the matrix

$$(25) \qquad \pi = \begin{pmatrix} 0 & e \\ p & 0 \end{pmatrix}$$

is $p(\lambda^2) = 0$, so that π has the single elementary divisor $p(\lambda^2)$ and the matrix

$$(26) \qquad N = \pi E + U$$

has the single elementary factor $[p(\lambda^2)]^k$. If

$$(27) \qquad X = \begin{bmatrix} 0 & 0 & \cdot & 0 & -\epsilon \\ 0 & 0 & \cdot & \epsilon & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & (-\epsilon)^{k-1} & \cdot & 0 & 0 \\ (-\epsilon)^k & 0 & \cdot & 0 & 0 \end{bmatrix},$$

it is easily shown that

$$XU = -U'X$$

and that, if

$$(28) \qquad V = q\tau X,$$

V is non-singular and satisfies (18). It should be noted that, since τ is skew symmetric, V is symmetric, if k is even, and skew symmetric if k is odd.

Type c. If, in (26), $\pi = 0$ and $\epsilon = 1$, so that U is the auxiliary unit matrix of order k,

$$(29) \qquad N = U,$$

is a matrix with the single elementary factor λ^k. Moreover the matrix V defined by (28) where q, τ and ϵ all have the value unity, satisfies (18).

If $N_1, N_2, \cdots, N_r$ are r matrices, where N_i is the matrix N with $k = k_i$, the diagonal block matrix

$$(30) \qquad M = [N_1, N_2, \cdots, N_r]$$

has the elementary factors $[p(\lambda)]^{k_i}$, $[p(-\lambda)]^{k_i}$, if each N_i is defined by (17); the elementary factors $[p(\lambda^2)]^{k_i}$, if each N_i is defined by (26); and the ele-

mentary factors λ^{k_i}, if each N_i is defined by (29) $(i = 1, 2, \cdots, r)$. Equation (24) or (28) with $k = k_i$ determines a non-singular matrix V_i such that $V_i N_i = - N' V_i$ and accordingly the matrix

$$Q = [V_1, V_2, \cdots, V_r] \tag{31}$$

is non-singular and satisfies (6), when M has the value given by (30).

Let the elementary factors of $A - \lambda B$ be $[p_i(\lambda)]^{k_{ij}}$, $[p_i(-\lambda)]^{k_{ij}}$ $(i = 1, 2, \cdots, s)$, of type a, $[p_i(\lambda^2)]^{k_{ij}}$ $(i = s + 1, \cdots, t - 1)$, of type b, $\lambda^{k_{tj}}$ of type c,

$$j = 1, 2, \cdots, r_i; \quad k_{i1} \geqq k_{i2} \geqq \cdots \geqq k_{ir_i},$$

where $p_i(\lambda) \neq p_j(\lambda)$, $p_i(\lambda^2) \neq p_j(\lambda^2)$, if $i \neq j$. Then the matrix

$$M = [M_1, M_2, \cdots, M_t], \tag{32}$$

where M_i is the matrix corresponding to the matrix on the left of (30), when p is replaced by p_i and r by r_i, has the same elementary factors and therefore the same invariant factors as $A - \lambda B$. Hence the matrix M is similar to the matrix AB^{-1}. Moreover, if Q_i is obtained from (31) in the same way as M_i is obtained from (30), the diagonal block matrix,

$$Q = [Q_1, Q_2, \cdots, Q_t], \tag{33}$$

satisfies the equation $QM = - M'Q$. We may accordingly take the matrices M and Q defined by (32) and (33) as the matrices M and Q of section 1. Since any matrix G, commutative with M in (32) is a diagonal block matrix[6] $[G_1, G_2, \cdots, G_t]$ by the remark at the end of section I we see that we may treat each block R_i separately.

3. *Reduction of R.* We consider the matrix $R_i = Q_i G_i$, where R_i is skew-symmetric and non-singular, G_i is commutative with M_i and $Q_i M_i = - M'_i Q_i$. We first treat the case, in which $i \leqq t - 1$, so that the elementary factors of M_i are not of the form λ^k. For simplicity of writing we temporarily drop all suffixes i and write R, Q, G, M, etc. for R_i, Q_i, G_i, M_i respectively so that M and Q are the matrices defined by equations (30) and (31) respectively.

If,

$$G = (G_{ij})$$

is a *partition* of G *similar* to that of M in (30), i. e. if G_{ij} is a matrix with

[6] R. C. Trott, *Bulletin of the American Mathematical Society*, January 1935, Abstract No. 95.

the same number of rows k_i as N_i and the same number of columns k_j as N_j, it is known that, when $k_i \geqq k_j$,

$$(34) \qquad G_{ij} = \begin{pmatrix} F_{ij} \\ 0 \end{pmatrix}, \qquad G_{ji} = (0 \ F_{ji}),$$

where F_{ij} and F_{ji} are square matrices of order $k = k_j$.[7] Moreover,

$$(35) \qquad F_{ij} = f_{ij0}E + f_{ij1}U + \cdots + f_{ij,k-1}U^{k-1},$$

where $f_{ijs} = f_{ijs}(\pi)$ is a polynomial in the matrix π with coefficients in K. Since $R = QG$ is skew-symmetric, it follows that

$$(36) \qquad V_iG_{ij} = -G'_{ji}V'_j = \rho_jG'_{ji}V_j,$$

where $\rho_j = +1$, if V_j is skew symmetric and $\rho_j = -1$ if V_j is symmetric. In particular, if $k_i = k_j$, since $G_{ij} = F_{ij}$ and $V_i = V_j$, (36) becomes

$$(37) \qquad V_iF_{ij} = \rho_jF'_{ji}V_j = \rho_iF'_{ji}V_i.$$

As a consequence of the definition of V_i,

$$V_i\pi U^a = -\pi'U'^aV_i.$$

Hence,

$$\begin{aligned} V_iF_{ij} &= V_i\sum_{s=0}^{k-1} f_{ijs}(\pi)U^s, \\ &= \sum_{s=0}^{k-1} f_{ijs}(-\pi')U'^sV_i, \\ &= \rho_i\sum_{s=0}^{k-1} f_{jis}(\pi')U'^sV_i \quad \text{by (37)}. \end{aligned}$$

Therefore, if $k_i = k_j$,

$$(38) \qquad f_{ijs}(-\pi) = \rho_if_{jis}(\pi) \qquad (s = 1, 2, \cdots, k-1).$$

In particular,

$$f_{iis}(-\pi) = \rho_if_{iis}(\pi).$$

Hence $f_{iis}(\pi)$ is an even polynomial in π if $\rho_i = 1$ and an odd polynomial, if $\rho_i = -1$. In either case $f_{iis}(\pi)$ is singular, if and only if it is zero. Consequently we have the result; *G_{ii} is singular, if and only if its first element f_{ii0} is zero.*

Let $k_1 = k_2 = \cdots = k_c > k_{c+1} \geqq k_{c+j}$. Then, if G_{11} is singular, but for some value of j, $1 < j \leqq c$, G_{jj} is non-singular, we may interchange the first and j-th rows, and the first and j-th columns, thus bringing G_{jj} into the place

[7] Trott, *loc. cit.*, cf. Cullis, *Matrices and Determinoids*, vol. 3, chap. XXVII.

of G_{11} without disturbing M or Q. We therefore suppose that G_{jj} is singular for all values of j, $1 \leqq j \leqq c$. Since the first element of G_{i1} is zero, when $i > c$, (equation (34)) and G is non-singular, the first element f_{j10} of G_{j1} is different from zero for at least one value of j, $1 < j \leqq c$. Accordingly without any loss of generality we may suppose that $f_{210} \neq 0$.

Let

$$W = \left[\begin{pmatrix} E_1 & 0 \\ w(\pi)E_1 & E_1 \end{pmatrix}, E_2\right],$$

where E_1 is the unit matrix of the same order as N_1 and E_2 the unit matrix of the same order as $[N_3, N_4, \cdots, N_r]$. The matrix W is commutative with M and

$$W'Q = \left[\begin{pmatrix} E_1 & w(\pi')E_1 \\ 0 & E_1 \end{pmatrix}, E_2\right]Q,$$

$$= Q\left[\begin{pmatrix} E_1 & w(-\pi)E_1 \\ 0 & E_1 \end{pmatrix}, E_2\right].$$

If $W'QGW = QH$, where $H = (H_{ij})$ is a partition of H similar to that of G,

$$H_{11} = G_{11} + w(-\pi)G_{21} + w(\pi)G_{12} + w(-\pi)w(\pi)G_{22}.$$

The first element h of H_{11} accordingly satisfies the equation

$$h = f_{110} + w(-\pi)f_{210} + w(\pi)f_{120} + w(-\pi)w(\pi)f_{220}$$

or

$$h = w(-\pi)f_{210} + w(\pi)f_{120},$$

since by hypothesis $f_{110} = f_{220} = 0$.

If $w(\pi)$ is the identity matrix h has the value $f_{210} + f_{120}$ and if $w(\pi) = \pi$, the value $\pi(f_{120} - f_{210})$. Since π is non-singular both these values of h cannot be zero, as otherwise f_{210} would be zero, contrary to our assumption. Thus we find a non-singular matrix W, such that $W'QG = QH$ where the first element of H_{11} is non-zero so that H_{11} is non-singular. We may therefore suppose that such a transformation has already been applied to G and accordingly may assume that G_{11} is non-singular.

The matrices Q and G now satisfy the hypothesis of Lemma 1, so that G may be reduced to a form, in which $G_{j1} = G_{1j} = 0$, $j \neq 1$. By $r-1$ repetitions of the above process we finally reduce G to the diagonal block matrix,

(39) $$G = [G_1, G_2, \cdots, G_r],$$

where $G_j = \sum_{s=0}^{k_j-1} \gamma_{js} U_j^s$ and $\gamma_{js} = \gamma_{js}(\pi)$ is a polynomial in the matrix π while γ_{j0} is non-singular.

We now proceed to reduce the matrix G_j to the form $\gamma_{j0}E_j$. Let

$$\gamma_{j1} = \gamma_{j2} = \cdots = \gamma_{jc} = 0, \quad \gamma_{j,c+1} \neq 0, \quad c \leqq k_{j-1}.$$

Then, if $W_j = E_j - \frac{1}{2}\gamma_{j0}^{-1}\gamma_{j\,c+1}U_j^{c+1}$,

$$W_j^2 G_j = H_j = \gamma_{j0}E_j + \sum_{s=c+2}^{k_j-1} h_{js}(\pi)U_j^s.$$

Moreover

$$\begin{aligned} W'_jV_j &= \{E_j - \tfrac{1}{2}(\gamma_{j0}^{-1}E_j)'(\gamma_{j\,c+1}U_j^{c+1})'\}V_j \\ &= \rho_j^2 V_j(E_j - \tfrac{1}{2}\gamma_{j0}^{-1}E_j\gamma_{j\,c+1}U_j^{c+1}), \text{ by (36)}, \\ &= V_jW_j. \end{aligned}$$

Therefore $$W'_jV_jG_jW_j = V_jW_j^2G_j = V_jH_j.$$

But H_j is of the same type as G_j, except that it contains no term in U_j^{c+1}. Accordingly, in at most $k_j - 1$ such steps, we may reduce G_j to the form

(40) $$G_j = \gamma_j E_j.$$

The matrix γ_j in (40) is a polynomial in π, which is even or odd according as V_j is skew symmetric or symmetric.

It is now necessary to distinguish between the two cases;

case a. $$\pi = \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix},$$

case b. $$\pi = \begin{pmatrix} 0 & e \\ p & 0 \end{pmatrix}.$$

In case *a* for all values of j, V_j is skew symmetric, so that γ_j in (40) is an even polynomial

$$\gamma_j = g_j(\pi^2) = \begin{pmatrix} g_j(p^2) & 0 \\ 0 & g_j(p^2) \end{pmatrix}.$$

Let $$r(\pi) = \begin{pmatrix} [g_j(p^2)]^{-1} & 0 \\ 0 & e \end{pmatrix}.$$

Then

$$\begin{aligned} r(\pi)'q\tau g_j(\pi^2)r(\pi) &= q\tau\begin{pmatrix} e & 0 \\ 0 & [g_j(p^2)]^{-1} \end{pmatrix}\begin{pmatrix} g_j(p^2) & 0 \\ 0 & g_j(p^2) \end{pmatrix}\begin{pmatrix} [g_j(p^2)]^{-1} & 0 \\ 0 & e \end{pmatrix}, \\ &= q\tau\begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \text{ by (21) and (22).} \end{aligned}$$

Therefore, if $W_j = r(\pi)E_j$,

$$W'_jV_jG_jW_j = V_j\begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}E_j = V_j = q\tau T_j.$$

Hence in case *a* we may reduce R and RM by transformations permutable with M, to the forms

$$(41) \qquad [q_T T_1, q_T T_2, \cdots, q_T T_r], \qquad [q_T T_1 N_1, q_T T_2 N_2, \cdots, q_T T_r N_r]$$

respectively. The matrices (41) are uniquely determined by the matrices p and q and by the exponents k_i to which $p(\lambda)$ and $p(-\lambda)$ occur among the elementary factors of $A - \lambda B$.

We condense the above results in the following statement: *Corresponding to each pair of elementary factors $[p(\lambda)]^k$, $[p(-\lambda)]^k$ of type a in the canonical form of $A - \lambda B$, there is a block*

$$(42) \qquad VN - \lambda V,$$

where N is defined by (17) *and V by* (24).

case b. In this case no such great simplification is in general possible. Let

$$G = [\gamma_1 E_1, \gamma_2 E_2, \cdots, \gamma_r E_r], \qquad H = [\sigma_1 E_1, \sigma_2 E_2, \cdots, \sigma_r E_r],$$

where σ_i and γ_i are all non-singular polynomials in π. Further let W be a non-singular matrix commutative with M such that,

$$(43) \qquad W'QGW = QH.$$

If, as in previous cases, we write $W = (W_{ij})$ as a matrix of matrices, it is a consequence of (43) that

$$(44) \qquad \sum_{a=1}^{r} W'_{ai} V_a \gamma_a E_a W_{aj} = \delta_{ij} V_i \sigma_i E_i \qquad (i, j = 1, 2, \cdots, r),$$

δ_{ij} the Kronecker δ.

Since W is commutative with M

$$W_{ij} = \begin{pmatrix} F_{ij} \\ 0 \end{pmatrix}, \quad W_{ji} = (0\ F_{ji}), \quad i \leqq j \quad \text{i. e.} \quad k_i \geqq k_j,$$

where F_{ij} is defined by (35). Moreover,

$$\begin{aligned} F'_{ij} V_j &= \{f_{ij0}(\pi') E_j + \sum_{s=1}^{k_j-1} f_{ijs}(\pi') U'^s_j\} V_j \\ &= V_j \{f_{ij0}(-\pi) E_j + \sum_{s=1}^{k_j-1} (-1)^s f_{ijs}(-\pi) U_j^s\} \\ &= V_j \tilde{F}_{ij}. \end{aligned}$$

Accordingly if $i \leqq j$,

$$W'_{ij} V_i = (F'_{ij}\ 0) V_i = (0\ F'_{ij} V_j) = (0\ V_j \tilde{F}_{ij}) = V_j (0\ \tilde{F}_{ij})$$

and

$$W'_{ji}V_j = \begin{pmatrix} 0 \\ F'_{ji} \end{pmatrix} V_j = \begin{pmatrix} 0 \\ F'_{ji}V_j \end{pmatrix} = \begin{pmatrix} 0 \\ V_j\bar{F}_{ji} \end{pmatrix} = V_i \begin{pmatrix} \bar{F}_{ji} \\ 0 \end{pmatrix}.$$

Hence,

(45) $$W'_{ij}V_i = V_j\bar{W}_{ij} \quad \text{and} \quad W'_{ji}V_j = V_i\bar{W}_{ji}$$

where

$$\bar{W}_{ij} = (0\ \bar{F}_{ij}) \quad \text{and} \quad \bar{W}_{ji} = \begin{pmatrix} \bar{F}_{ji} \\ 0 \end{pmatrix} \quad k_i \geqq k_j.$$

Therefore, if $w_{ij}(\pi)$ and $\bar{w}_{ij}(\pi)$ are the first elements of the matrices W_{ij} and $\bar{W}_{ij}$ respectively we have the results

(46) $$\bar{w}_{ij}(\pi) = w_{ij}(-\pi), k_i = k_j;\ \bar{w}_{ij}(\pi) = 0, k_i > k_j;\ w_{ij}(\pi) = 0, k_i < k_j.$$

It follows from (44) and (45) that

$$V_i \sum_{a=1}^{r} \bar{W}_{ai}\gamma_a E_a W_{aj} = \delta_{ij} V_i \sigma_i E_i,$$

or since V_i is non-singular that

(47) $$\sum_{a=1}^{r} \bar{W}_{ai}\gamma_a E_a W_{aj} = \delta_{ij}\sigma_i E_i \qquad (i, j = 1, 2, \cdots, r).$$

As a consequence of the nature of the matrices $\bar{W}_{ai}$ and W_{aj}, (47) remains true when each matrix is replaced by its first element, so that

$$\sum_{a=1}^{r} \bar{w}_{ai}(\pi)\gamma_a w_{aj}(\pi) = \delta_{ij}\sigma_i \qquad (i, j = 1, 2, \cdots, r).$$

If $k_{c-1} > k_c = k_{c+1} = \cdots = k_d > k_{d+1}$, it follows from (46) and the last equation that

(48) $$\sum_{a=c}^{d} w_{ai}(-\pi)\gamma_a w_{aj}(\pi) = \sigma_i\delta_{ij}, \qquad c \leqq i \leqq d, \quad c \leqq j \leqq d.$$

The matrices γ_i and σ_i $(i = c, c+1, \cdots, d)$, are either all even polynomials in π or else all odd polynomials in π. We may therefore write

(49) $$\gamma_i = g_i(\pi^2)\pi^a, \qquad \sigma_i = h_i(\pi^2)\pi^a \qquad (a = 0 \text{ or } 1),$$

so that (48) becomes,

(50) $$\sum_{a=c}^{d} w_{ai}(-\pi) g_a(\pi^2) w_{aj}(\pi) = h_i(\pi^2)\delta_{ij}, \qquad c \leqq i \leqq d, \quad c \leqq j \leqq d.$$

If θ^2 is a zero of $p(x)$, the field $K_1 = K(\theta^2)$ is simply isomorphic to the

field of all polynomials in π^2 with coefficients in K and the field $K_2 = K(\theta)$ is simply isomorphic to the field of all polynomials in π with coefficients in K. Accordingly (50) implies

$$(51) \qquad \sum_{a=c}^{d} w_{ai}(-\theta) g_a(\theta^2) w_{aj}(\theta) = h_i(\theta^2)\delta_{ij},$$

and conversely (51) implies (50). The field K_2 is quadratic over K_1 and, if $w(\theta)$ is an element of K_2, $w(-\theta) = \bar{w}$, is its conjugate. Hence, if

$$C = (c_{ij}) \qquad (i, j = 1, 2, \cdots, d+1-c),$$

where

$$c_{ij} = w_{c+i-1, c+j-1}(\theta),$$

(51) is equivalent to

$$(52) \qquad C^*[g_c(\theta^2), g_{c+1}(\theta^2), \cdots, g_d(\theta^2)]\, C = [h_c(\theta^2), \cdots, h_d(\theta^2)],$$

where C^* is the conjugate transposed of C.

By a suitable interchange of rows and columns it can be shown that $| w_{ij}(\pi) |$, $c \leqq i$, $j \leqq d$ is a factor of $| W |$. Hence, since W is non-singular, $| w_{ij}(\pi) | \neq 0$ and therefore $| C | \neq 0$, so that C is non-singular. Hence, *the two matrices* $[g_c(\theta^2), g_{c+1}(\theta^2), \cdots, g_d(\theta^2)]$ *and* $[h_c(\theta^2), h_{c+1}(\theta^2), \cdots, h_d(\theta^2)]$ *with elements in* K_2 *are equivalent under a non-singular conjunctive transformation with elements in* K_1.

Conversely, if in (52) θ is replaced by π, we have

$$\tilde{C}(\pi)[g_c(\pi^2), \cdots, g_d(\pi^2)]C(\pi) = [h_c(\pi^2), \cdots, h_d(\pi^2)]$$

and, if W_c is the direct product of $C(\pi)$ and the unit matrix E_c,

$$W_c[g_c(\pi^2)E_c, \cdots, g_d(\pi^2)E_d]\, W_c = [h_c(\pi^2)E_c, \cdots, h_d(\pi^2)E_d].$$

Multiplying this last equation by π^a and using (49) we get,

$$(53) \qquad \tilde{W}_c[\gamma_c E_c, \cdots, \gamma_d E_d]\, W_c = [\sigma_c E_c, \cdots, \sigma_d E_d].$$

Equation (43) implies a set of equations (53), one for each distinct value of the exponents k_i of $p(\lambda^2)$. If $W = [W_1, W_2, \cdots, W_f]$ and $W_1, W_2, \cdots, W_f$ are the matrices W_c of (52) corresponding to the distinct equations of the set (53),

$$\tilde{W}GW = H \text{ or } Q\tilde{W}GW = QH. \quad \text{Since } Q\tilde{W} = W'Q,$$

it follows that $W'QGW = QH$ and, since each matrix W_c is non-singular,

that W is non-singular. Hence in case b we may reduce R and RM by transformations permutable with M to the forms,

$$(54) \qquad [q\tau\gamma_1 X_1, \cdot \cdot \cdot, q\tau\gamma_r X_r] \quad \text{and} \quad [q\tau\gamma_1 X_1 N_1, \cdot \cdot \cdot, q\tau\gamma_r X_r N_r]$$

respectively where $\gamma_i = g_i(\pi^2)$, if k_i is odd and $\gamma_i = \pi g_i(\pi^2)$, if k_i is even. The matrices (54) are not uniquely determined by the matrices p, q and the exponents k_i of $p(\lambda^2)$. We may express the results as follows: *If* $[p(\lambda^2)]^k$ *occurs exactly* a *times among the elementary factors of the pencil* $A - \lambda B$, *corresponding to* $[p(\lambda^2]^k$ *in the canonical form there is a block*

$$(55) \qquad [V\gamma_1 N - \lambda V\gamma_1, \cdot \cdot \cdot, V\gamma_a N - \lambda V\gamma_a]$$

where N *is defined by* (26) *and* V *by* (28). *With this block, and so with* $[p(\lambda^2)]^k$, *is associated a diagonal matrix of order* a *with elements in the field* $K(\theta^2)$, *where* θ^2 *is a zero of* $p(x) = 0$. *This associated matrix is not uniquely determined but is determined apart from a non-singular conjunctive transformation in the field* $K(\theta)$.

If K is the field of all real numbers the only irreducible even polynomials are of the type $p(\lambda^2) = \lambda^2 + b^2$. Hence $K(\theta^2) = K$ and $K(\theta)$ is the field of all complex numbers. Since any real quadratic form is equivalent in the real field to a sum of a certain number of positive and negative squares, the matrix associated with an elementary factor $(\lambda^2 + b^2)^k$ may be reduced to the simple form $[\rho_1, \rho_2, \cdot \cdot \cdot, \rho_a]$ where $\rho_i = +1 \;\; i \leqq d$, $\rho_i = -1 \;\; i > d$, and d is uniquely determined. In fact d is the index of the quadratic form.

Case c. The reduction in this case is similar in many respects to that of the previous cases. Equations (34) and (35) are true, where f_{ijs} is now an ordinary number and no longer a matrix. We assume that

$$k_1 = k_2 = \cdot \cdot \cdot = k_c > k_{c+j}.$$

If k_1 is even V_1 is skew symmetric and, by a proof exactly similar to that in case a or b, we can reduce the matrix (G_{ij}) $(i, j = 1, 2, \cdot \cdot \cdot, c)$, to a diagonal matrix $[g_1 E_1, g_2 E_2, \cdot \cdot \cdot, g_c E_c]$ where, as in case b, the diagonal matrix $[g_1, g_2, \cdot \cdot \cdot, g_c]$ is only determined to within a non-singular congruent transformation with elements in K. Therefore corresponding to an elementary factor λ^{2k} there is in the canonical form of $A - \lambda B$ a block

$$gX_{2k}U_{2k} - \lambda g X_{2k},$$

where $g \neq 0$ and K_{2k} is defined by (27) with $\epsilon = 1$, while U_{2k} is the auxiliary unit matrix of order $2k$. On rearranging the rows and columns of X_{2k} and

U_{2k} in the order $1, 3, \cdots, 2k-1, 2, 4, \cdots, 2k$ we find that X_{2k} and U_{2k} are equivalent respectively to

$$\begin{pmatrix} 0 & -q \\ q & 0 \end{pmatrix} = q\tau \quad \text{and} \quad \pi = \begin{pmatrix} 0 & e \\ p & 0 \end{pmatrix}$$

where e is the unit matrix of order k and p is the auxiliary unit matrix of order k while q, which satisfies (21), is the matrix T of (19) when $\epsilon = 1$. Hence *an elementary factor* λ^{2k} *may be considered to be of type b where* $p(\lambda) = \lambda^{2k}$ *and* $p = U$.

If k_1 is odd, V_1 is symmetric and $f_{110} = 0$, so that G_{11} is singular. As in previous cases we may suppose that $f_{210} \neq 0$ and since $f_{120} = -f_{210}$, it is easily shown that $\begin{vmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{vmatrix} \neq 0$. By repeated applications of Lemma 1 it therefore follows that c must be even and that G may be reduced to the[8] diagonal block form

$$[H_1, H_2, \cdots, H_{c/2}],$$

where H_j is a square matrix of $2k_1$ rows of the same type as $H_1 = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$. It is not possible to reduce the matrix H_1, for example, to diagonal form. Accordingly we proceed as follows and consider the pencil

$$(56) \qquad \begin{pmatrix} V_1 & 0 \\ 0 & V_1 \end{pmatrix} H_1 \left\{ \begin{pmatrix} U_1 & 0 \\ 0 & U_1 \end{pmatrix} - \lambda \begin{pmatrix} E_1 & 0 \\ 0 & E_1 \end{pmatrix} \right\}.$$

The elementary factors of this pencil are $\lambda^{k_1}, \lambda^{k_1}$. But with the notation of (15) the elementary factors of $\phi U_1 - \lambda \epsilon E_1$ are also $\lambda^{k_1}, \lambda^{k_1}$, if $e = 1$. Hence the pencil (56) is equivalent to

$$VG(N - \lambda E),$$

where $N = \phi U_1$ and V is defined by (24) with $e = 1$ and $q = 1$. The matrix G is permutable with N and, if G is considered as a matrix of two rowed matrices, every element to the left of the leading diagonal is zero so that,

$$G = G_0 + G_1 + G_2 + \cdots + G_{k-1},$$

where G_i is the matrix formed by the elements of G in the i-th diagonal to the right of the leading one. Moreover, since G is non-singular G_0 is non-singular. Further $G_i N = N G_i$ and $VG_i = -G'_i V' = G'_i V$. If $G_1 = G_2 = G_{f-1} = 0$ and $G_f \neq 0$, the matrix $W = E - 1/2 G_0^{-1} G_f$ is permutable with N and satisfies the equation

$$W'VGW = V(G_0 + H_{f+1} + \cdots H_{k-1})$$

[8] This is a well known result. See Turnbull and Aitken, *Canonical Matrices*, p. 137.

where H_j is of the same type as G_j. Hence we may assume that $G_1 = G_2 = G_{k-1} = 0$. Since G_0 is commutative with N and VG_0 is skew symmetric,

$$G_0 = g\,\epsilon\,E_1,$$

where g is an element of K. A reduction similar to that in case b shows that g may be taken as $+1$.

Hence, *if λ^k occurs among the elementary factors of $A - \lambda B$ and k is odd it must occur an even number, $2a$, of times. In the canonical form of $A - \lambda B$ occur a blocks of the nature*

$$\tau T(\phi U - \lambda\,\epsilon\,E).$$

It should be noted that the two elementary factors λ^k, λ^k are accordingly of type (a) where $p(\lambda) = \lambda$ and π is the zero matrix. We have accordingly proved the theorem

THEOREM I. *A canonical form for the pencil $A - \lambda B$, where A is symmetric and B is skew symmetric, under non-singular congruent transformation in K, is a diagonal block matrix, whose component blocks are given by equation* (42) *or equation* (55).

COROLLARY. *Necessary and sufficient conditions that two such pencils $A - \lambda B$ and $C - \lambda D$ be equivalent in K are that,*

(a) *the elementary factors of $A - \lambda B$ be the same as those of $C - \lambda D$.*

(b) *the matrix associated with each elementary factor of the type $[p(\lambda^2)]^k$ in a normal form of $A - \lambda B$ be equivalent under a conjunctive transformation to the corresponding matrix in a normal form of $C - \lambda D$.*

4. *Reduction of B.* Since B is non-singular and skew symmetric there exists a non-singular matrix P with elements in K such that

$$P'BP = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}, \tag{57}$$

where E is the unit matrix of order one half the order of B. We now proceed to find a canonical form for the pencil $A - \lambda B$, in which B is equivalent to the simple matrix on the right of (57). We start with the canonical form deduced in the previous sections and have in all to consider three cases.

Case a. Corresponding to the elementary factors $[p(\lambda)]^k$, $[p(-\lambda)]^k$ in the canonical form is the block $q\tau TN - \lambda q\tau T$ (equation (42)), where

$$\tau T = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & -e \\ 0 & 0 & 0 & 0 & \cdot & e & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & -e & \cdot & 0 & 0 \\ 0 & 0 & e & 0 & \cdot & 0 & 0 \\ 0 & -e & 0 & 0 & \cdot & 0 & 0 \\ e & 0 & 0 & 0 & \cdot & 0 & 0 \end{bmatrix}.$$

By rearranging the rows and columns of τT in the order $1, 3, 5, \cdots, 2k-1, 2, 4, \cdots, 2k$, we see that $\tau T \approx \begin{pmatrix} 0 & -T_e \\ T_e & 0 \end{pmatrix}$ [9] where T_e is the matrix

$$\begin{bmatrix} 0 & 0 & \cdot & e \\ \cdot & \cdot & \cdot & \cdot \\ 0 & e & \cdot & 0 \\ e & 0 & \cdot & 0 \end{bmatrix}.$$

The same transformation applied to N shows that

$$N \approx \begin{pmatrix} L & 0 \\ 0 & -L \end{pmatrix},$$

where

$$(58) \qquad L = \begin{bmatrix} p & e & \cdot & 0 & 0 \\ 0 & p & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & p & e \\ 0 & 0 & \cdot & 0 & p \end{bmatrix}.$$

If $W = \begin{pmatrix} q^{-1}T_e & 0 \\ 0 & E \end{pmatrix}$,

$$W'q\begin{pmatrix} 0 & -T_e \\ T_e & 0 \end{pmatrix} W = \begin{pmatrix} q^{-1}T_e & 0 \\ 0 & E \end{pmatrix}\begin{pmatrix} 0 & -qT_e \\ qT_e & 0 \end{pmatrix}\begin{pmatrix} q^{-1}T_e & 0 \\ 0 & E \end{pmatrix} = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}.$$

Further

$$W'q\begin{pmatrix} 0 & -T_e \\ T_e & 0 \end{pmatrix}\begin{pmatrix} L & 0 \\ 0 & -L \end{pmatrix} W = \begin{pmatrix} 0 & L \\ qT_eLq^{-1}T_e & 0 \end{pmatrix} = \begin{pmatrix} 0 & L \\ L' & 0 \end{pmatrix},$$

since $qT_eL = L'qT_e$. Hence

$$q\tau TN - \lambda q\tau T \approx \begin{pmatrix} 0 & L \\ L' & 0 \end{pmatrix} - \lambda\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}.$$

It accordingly follows that, if $R_aM_a - \lambda R_a$ is the part of the canonical form of $A - \lambda B$ depending on elementary factors of type a, including those of type λ^k where k is odd,

$$(59) \qquad R_aM_a - \lambda R_a \approx \begin{pmatrix} 0 & F \\ F' & 0 \end{pmatrix} - \lambda\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}.$$

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[9] We use $\approx$ to denote "is equivalent to."

where

(60) $$F = [L_1, L_2, \cdots, L_w]$$

is a diagonal block matrix, which is the direct sum of all matrices L defined by (58), one for each pair of elementary factors $[p(\lambda)]^k$, $[p(-\lambda)]^k$ of type a.

case b. Corresponding to the elementary factor $[p(\lambda^2)]^k$ in the canonical form is the block $q\tau\gamma XN - \lambda q\tau\gamma X$ (equation 55). It is necessary to consider the cases in which k is even and in which k is odd separately. If $k = 2f$ is even, the matrix $V = q\tau X$ is skew symmetric, so that $\gamma = g(\pi^2)\pi$ is an odd polynomial in π. Hence

(61) $$q\tau\gamma = -\gamma' q\tau.$$

By rearranging the rows and columns in the order $1, 3, \cdots, 2f-1, 2, 4, \cdots, 2f$ we find that

$$X \approx \begin{pmatrix} 0 & -T \\ T & 0 \end{pmatrix} \quad \text{and} \quad N \approx \begin{pmatrix} \pi E & E \\ U & \pi E \end{pmatrix}$$

where T is defined by (19) and E and U by (17).

If $\psi = q\tau\gamma$,

$$\psi' = \gamma' q' \tau' = -\gamma' q\tau = q\tau\gamma \text{ by (61)},$$

so that ψ is symmetric. Accordingly, if $W = \begin{pmatrix} E & 0 \\ 0 & \psi^{-1}T \end{pmatrix}$,

$$W' q\tau\gamma X W \approx \begin{pmatrix} E & 0 \\ 0 & T\psi^{-1} \end{pmatrix} \begin{pmatrix} 0 & -\psi T \\ \psi T & 0 \end{pmatrix} \begin{pmatrix} E & 0 \\ 0 & \psi^{-1}T \end{pmatrix} = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}.$$

Similarly,

$$W' q\tau\gamma XNW \approx \begin{pmatrix} E & 0 \\ 0 & \psi^{-1}T \end{pmatrix} \begin{pmatrix} 0 & -\psi T \\ \psi T & 0 \end{pmatrix} \begin{pmatrix} \pi E & E \\ U & \pi E \end{pmatrix} \begin{pmatrix} E & 0 \\ 0 & \psi^{-1}T \end{pmatrix},$$

$$\approx \begin{pmatrix} -\psi TU & -\psi T\pi\psi^{-1}T \\ \pi E & \psi^{-1}T \end{pmatrix}.$$

Hence

(62) $$q\tau\gamma XN \approx Z = \begin{pmatrix} -\psi TU & \pi' E \\ \pi E & \psi^{-1}T \end{pmatrix}.$$

For example if $k = 6$,

$$Z = \begin{bmatrix} 0 & 0 & 0 & \pi' & 0 & 0 \\ 0 & 0 & -\psi & 0 & \pi' & 0 \\ 0 & -\psi & 0 & 0 & 0 & \pi' \\ \pi & 0 & 0 & 0 & 0 & \psi^{-1} \\ 0 & \pi & 0 & 0 & \psi^{-1} & 0 \\ 0 & 0 & \pi & \psi^{-1} & 0 & 0 \end{bmatrix},$$

where $\pi = \begin{pmatrix} 0 & e \\ p & 0 \end{pmatrix}$ and $\psi = \begin{pmatrix} -qpg & 0 \\ 0 & qg \end{pmatrix}$ and $g = g(p)$ is a polynomial in p.

We may write the matrix Z of (62) in the form $Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z'_{12} & Z_{22} \end{pmatrix}$, where $Z_{11} = -\psi TU$ etc. Then if $R_{b_1}M_{b_1} - \lambda R_{b_1}$ is that part of the canonical form of $A - \lambda B$ depending on elementary factors of type b, i. e. on $[p(\lambda^2)]^k$, where k is even,

(63) $$R_{b_1}M_{b_1} - \lambda R_{b_1} \approx \begin{pmatrix} Y_{11} & Y_{12} \\ Y'_{12} & Y_{22} \end{pmatrix} - \lambda \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix},$$

where

(64) $$Y_{ij} = [Z_{ij,1}, Z_{ij,2}, \cdots, Z_{ij,w}] \qquad (i, j = 1, 2),$$

is a diagonal block matrix, which is the direct sum of all matrices $Z_{ij,r}$ one for each elementary factor $[p(\lambda^2)]^k$, k even.

If however k is odd, the matrix V in (55) is symmetric so that $\gamma = g(\pi^2)$ is an even polynomial in π. In fact $q\tau\gamma = \begin{pmatrix} 0 & -q \\ q & 0 \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$, where $g = g(p)$ is a polynomial in p and

$$q\tau\gamma X = \begin{bmatrix} 0 & 0 & \cdot & 0 & 0 & 0 & qg \\ 0 & 0 & \cdot & 0 & 0 & -qg & 0 \\ 0 & 0 & \cdot & 0 & -qg & 0 & 0 \\ 0 & 0 & \cdot & qg & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & qg & \cdot & 0 & 0 & 0 & 0 \\ -qg & 0 & \cdot & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Rearranging the rows and columns of this matrix in the order 1, 3, 5, etc., we find

$$q\tau\gamma X \approx \begin{pmatrix} 0 & -qgX_e \\ qgX_e & 0 \end{pmatrix},$$

where X_e is symmetric, and is defined by (27) with ϵ replaced by e. The same transformation shows that $N \approx \begin{pmatrix} U & E \\ pE & U \end{pmatrix}$. If $W = \begin{pmatrix} (qg)^{-1}X_e & 0 \\ 0 & E \end{pmatrix}$,

$$W'q\tau\gamma XW \approx \begin{pmatrix} X_e(qg)^{-1} & 0 \\ 0 & E \end{pmatrix}\begin{pmatrix} 0 & -qgX_e \\ qgX_e & 0 \end{pmatrix}\begin{pmatrix} X_e(qg)^{-1} & 0 \\ 0 & E \end{pmatrix} = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix},$$

while

(65) $$W'q\tau\gamma XNW \approx \begin{pmatrix} -p(qg)^{-1}X_e & -U \\ -U' & qgX_e \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = C.$$

For example, if $k = 3$,

$$C = \begin{pmatrix} 0 & 0 & p(qg)^{-1} & 0 & -e & 0 \\ 0 & -p(qg)^{-1} & 0 & 0 & 0 & -e \\ p(qg)^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -qg \\ -e & 0 & 0 & 0 & qg & 0 \\ 0 & -e & 0 & -qg & 0 & 0 \end{pmatrix}.$$

Accordingly, if $R_{b_2}M_{b_2} - \lambda R_{b_2}$ is that part of the canonical form of $A - \lambda B$ depending on elementary factors of type b, i. e. on $[p(\lambda^2)]^k$, where k is odd,

$$(66) \qquad R_{b_2}M_{b_2} - \lambda R_{b_2} \approx \begin{pmatrix} D_{11} & D_{12} \\ D_{12} & D_{22} \end{pmatrix} - \lambda \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix},$$

where

$$(67) \qquad D_{ij} = [C_{ij,1}, C_{ij,2}, \cdots, C_{ij,w}] \qquad (i, j = 1, 2),$$

is a diagonal block matrix which is the direct sum of all matrices C_{ij} one for each elementary factor $[p(\lambda^2)]^k$, k odd (including the case $\lambda^{2k} = p(\lambda^2)$). It is an immediate consequence of equations (59), (63) and (64) that

THEOREM 2. *The pencil $A - \lambda B$ is equivalent in K to the pencil*

$$A_1 - \lambda \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$$

where

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & F & 0 & 0 \\ 0 & Y_{11} & 0 & 0 & Y_{12} & 0 \\ 0 & 0 & D_{11} & 0 & 0 & D_{12} \\ F' & 0 & 0 & 0 & 0 & 0 \\ 0 & Y'_{12} & 0 & 0 & Y_{22} & 0 \\ 0 & 0 & D'_{12} & 0 & 0 & D_{22} \end{pmatrix},$$

and F, Y_{ij}, D_{ij} are defined by (60), (64) and (67) respectively.

COROLLARY. *The symmetric matrix A is equivalent to the matrix A_1 under a non-singular congruent transformation in K, which leaves the skew symmetric matrix $\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$ invariant.*

This corollary follows immediately by substituting the matrix $\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$ for B in the pencil $A - \lambda B$.

5. *K the field of all real numbers.* In the canonical forms of the previous sections occur the matrices p and q, where p is any matrix with elements in K

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whose characteristic equation is $p(\lambda) = 0$, and q is a non-singular symmetric matrix satisfying the equation $qp = p'q$. If p is chosen as the companion matrix of $p(\lambda) = 0$, a comparatively simple matrix q can be determined.[10] If however K is the field of all real numbers, there are only three possible types for the irreducible equation $p(\lambda)$, and the corresponding values of p and q are even more simple. These are

$$(1)\ p(\lambda) = \lambda - a;\ (2)\ p(\lambda) = \lambda^2 - 2a\lambda + a^2 + b^2;\ (3)\ p(\lambda^2) = \lambda^2 + a^2.$$

In case (1) $p = a,\ q = 1$;

In case (2) $p = \begin{pmatrix} a & b \\ -b & a \end{pmatrix},\ q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;

In case (3) $p = -a^2,\ q = 1,\ \pi = \begin{pmatrix} 0 & 1 \\ -a^2 & 0 \end{pmatrix}$.

Moreover each matrix g occurring in Y_{ij} or D_{ij} (equations (64) and (67)) now has the value ± 1.

The matrix $p = -a^2$, in case (3), is obtained by particularizing the general formula but for some purposes it is preferable to take $\pi = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$ instead of $\begin{pmatrix} 0 & 1 \\ -a^2 & 0 \end{pmatrix}$. If this is done, it is easily seen, that the matrix Z in (62) is unaltered, except that $\psi = \begin{pmatrix} ga & 0 \\ 0 & ga \end{pmatrix}$ where g is a real number. Since, in ψ, g may be replaced by any real number with the same sign, we may take $\psi = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}$ where $\rho = \pm 1$. Similarly the matrix C in (65) is only altered to the extent that

$$C_{11} = \rho a X_2; \qquad C_{22} = \rho a X_2.$$

In conclusion we exhibit the possible canonical forms, to one of which a real symmetric matrix A of order 4 can be reduced by a real non-singular congruent transformation, which leaves invariant the skew symmetric matrix $\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$. If A is non-singular, the possible elementary divisors of $A - \lambda \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$ are

$$(\alpha)\ (\lambda \pm a), (\lambda \pm b);\quad (\beta)\ (\lambda \pm a \pm ib);\quad (\gamma)\ (\lambda \pm ia), (\lambda \pm ib);$$
$$(\delta)\ (\lambda \pm a), (\lambda \pm ib);\quad (\epsilon)\ (\lambda \pm a)^2;\quad (\zeta)\ (\lambda \pm ia)^2.$$

The corresponding canonical forms for A are

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[10] J. Williamson, *loc. cit.*, p. 490.

$$(\alpha)\quad \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{pmatrix}; \qquad (\beta)\quad \begin{pmatrix} 0 & 0 & a & -b \\ 0 & 0 & b & a \\ a & b & 0 & 0 \\ -b & a & 0 & 0 \end{pmatrix};$$

$$(\gamma)\quad \begin{pmatrix} \rho a & 0 & 0 & 0 \\ 0 & \sigma b & 0 & 0 \\ 0 & 0 & \rho a & 0 \\ 0 & 0 & 0 & \sigma b \end{pmatrix} \rho, \sigma = \pm 1; \qquad (\delta)\quad \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & \rho b & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho b \end{pmatrix}, \rho = \pm 1;$$

$$(\epsilon)\quad \begin{pmatrix} 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \\ a & 0 & 0 & 0 \\ 1 & a & 0 & 0 \end{pmatrix}; \qquad (\zeta)\quad \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & \rho & 0 \\ a & 0 & 0 & \rho \end{pmatrix}, \rho = \pm 1.$$

The matrices in cases (α), (β), (ϵ) depend solely on the elementary divisors of the pencil; (γ) yields 4 or 3 non-equivalent matrices according as a is not or is the same as b while (δ) and (ζ) both yield two non-equivalent matrices.

If A is singular and the pencil has the pair of elementary divisors λ, λ the canonical form is obtained from (α) or (δ) by putting $a = 0$ and from (α) by putting $a = b = 0$ if the pencil contains the 4 elementary divisors $\lambda, \lambda, \lambda, \lambda$. If λ^2 occurs among the elementary divisors the canonical form is obtained from that corresponding to $(\lambda \pm ia)$ by replacing the first a by unity and the other by zero. If λ^4 is an elementary divisor the canonical form is

$$\begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\rho \\ 0 & 0 & -\rho & 0 \end{pmatrix}, \rho = \pm 1.$$

Thus we have determined a complete list of the possible canonical forms for the case $n = 4$.

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ON THE MOMENTUM PROBLEM FOR DISTRIBUTION FUNCTIONS IN MORE THAN ONE DIMENSION. II.

By E. K. Haviland.

It has recently been proved in a paper [1] which will be referred to as I and which is based on an extension of a method of M. Riesz that for the existence of a distribution function solving the momentum problem corresponding to a given n-dimensional matrix $\| c_{k_1 \ldots k_n} \|$ it is necessary and sufficient that the matrix be non-negative in the sense that if

$$P(x_1, \cdots, x_n) = \sum_{k_1=0}^{N_1} \cdots \sum_{k_n=0}^{N_n} a_{k_1 \ldots k_n} x_1^{k_1} \cdots x_n^{k_n}$$

be a polynomial non-negative for all $(x_1, \cdots, x_n)$, the corresponding functional value

$$P_c = \sum_{k_1=0}^{N_1} \cdots \sum_{k_n=0}^{N_n} a_{k_1 \ldots k_n} c_{k_1 \ldots k_n}$$

is likewise non-negative. A. Wintner has subsequently suggested that it should be possible to extend this result by requiring that the distribution function solving the problem have a spectrum contained in a preassigned set, a result which would show the well-known criteria for the various standard special momentum problems (Stieltjes, Herglotz, Hamburger, Hausdorff in one or more dimensions) to be but particular cases of the general n-dimensional momentum problem mentioned above. The purpose of the present note is to carry out this extension. It turns out that this unified and more general treatment of all these momentum problems is by no means more complicated than the several individual treatments to be found in the literature for these special cases. As in I, the proofs are given, for convenience, in the case of two dimensions.

The general result is given by the

Theorem. *For the existence of a distribution function $\phi(E)$ whose spectrum S is contained in a given set C of the plane and which is such that*

$$(1) \qquad \int\int_T x^n y^m d_{xy}\phi(E) = c_{nm}, \qquad (n, m = 0, 1, 2, \cdots; c_{00} = 1),$$

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[1] E. K. Haviland, "On the momentum problem for distribution functions in more than one dimension," *American Journal of Mathematics*, vol. 57 (1935), pp. 562-572. Further references to the momentum problem are to be found in this paper.

where T denotes the entire (x, y)-plane, it is necessary and sufficient that to any polynomial

$$P(x, y) = \sum_{n=0}^{N} \sum_{m=0}^{M} a_{nm} x^n y^m,$$

non-negative for all points (x, y) of C, there correspond the non-negative functional value

$$P_c = \sum_{n=0}^{N} \sum_{m=0}^{M} a_{nm} c_{nm}.$$

The matrix $\| c_{nm} \|$ will then be said to be non-negative with respect to C.

Proof. We may suppose C (which may be the entire (x, y)-plane) closed, since a spectrum is necessarily closed and since polynomials non-negative on a set C are non-negative on its closure $\bar{C}$ also.

The necessity of the condition is immediately clear. For C can then be taken to be the spectrum S itself. As S is closed, it belongs to the domain of definition of ϕ. Then $\int\int_T = \int\int_{T-S} + \int\int_S$. The second integral vanishes and if $P(x, y) \geq 0$ on S, the last integral is non-negative. Hence if (1) is to hold, we must have $P_c \geq 0$.

The proof of the existence of a distribution function $\phi(E)$ satisfying (1) under the hypotheses of the present theorem is effectively identical with the proof of the sufficient condition of the theorem in the paper I. One has only to replace the expression "non-negative" by the expression "non-negative with respect to C." The functions $g_{ij}(x, y)$ are defined as before, but in extending the functional operation to the modul generated by finite linear combinations of $1, x, y, xy, \cdots; g_{11}(x, y), \cdots, g_{ij}(x, y)$, one has to consider those elements of preceding moduls not less than (or not greater than) $g_{ij}(x, y)$ for any (x, y) *in* C. Since functions non-negative in T, the (x, y)-plane, are a fortiori non-negative on C, the proof then follows without further changes and the existence of such a $\phi(E)$ is assured. It remains only to be shown that the spectrum of ϕ is contained in C.

Let $P: (\xi, \eta)$ be a point of $T - C$. As C is closed, $T - C$ is open. Hence it is possible to find among the everywhere dense set of lines $x = \xi_i$, $y = \eta_j$ (the notation being that of the paper I) four forming the sides of a rectangle $R_1: (\xi_a \leq x < \xi_b; \eta_c \leq y < \eta_d)$ which contains (ξ, η) in its interior and is in turn contained in the interior of $T - C$. Consider the functions

$$H_1(x, y) = g_{bc}(x, y) - g_{ac}(x, y) \quad \text{and} \quad H_2(x, y) = g_{bd}(x, y) - g_{ad}(x, y).$$

There exists a modul such that both functions belong to it (and hence to all

succeeding moduls). To the former function corresponds the functional value $\gamma_{bc} - \gamma_{ac}$ and to the latter the functional value $\gamma_{bd} - \gamma_{ad}$. Both are non-negative and they can differ only if $H_1(x, y) \neq H_2(x, y)$ for some point (x, y) of C. As the two are identical for all points of C (for they are identical everywhere outside R_1), we have $\psi(R_1) = \gamma_{bd} - \gamma_{ad} - \gamma_{bc} + \gamma_{ac} = 0$. Hence if R_2 is a non-singular rectangle of ψ lying in R_1 and containing (ξ, η), we see $\psi(R_2) = \phi(R_2) = 0$. Consequently,[2] (ξ, η), which was any point of $T - C$, does not belong to the spectrum of ϕ, i. e., the spectrum of ϕ is contained in C, q. e. d.

We shall now examine the criteria for the solubility of the various standard special momentum problems and shall show these to be particular cases of the general criterion contained in our theorem.

1. *The one-dimensional Hamburger problem.* It is known[3] that every polynomial $P(x, y)$ with real coefficients which is non-negative for all points on the x-axis can be written in the form

$$[A(x)]^2 + [B(x)]^2 + yF(x, y),$$

where $A(x)$, $B(x)$ and $F(x, y)$ are polynomials with real coefficients. To this will correspond by (1), the set C being now the real axis, a functional value of the form

$$\sum_{h=0}^{n} \sum_{k=0}^{n} a_h a_k d_{h+k} + \sum_{h=0}^{m} \sum_{k=0}^{m} b_h b_k d_{h+k},$$

since $c_{ij} = 0$, $j \neq 0$. Here, as in what follows, c_{n0} is denoted by d_n. This functional value will be non-negative if and only if every section of the matrix $\| d_{h+k} \|$, $(h, k = 0, 1, 2, \cdots)$, belongs to a non-negative definite quadratic form. This is the criterion for the solubility of the one-dimensional Hamburger momentum problem.

2. *The Stieltjes problem.* Again, since[4] every real polynomial $P(x, y)$ which is non-negative for all non-negative points on the x-axis is of the form

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[2] Cf. E. K. Haviland, "On the theory of absolutely additive distribution functions," *American Journal of Mathematics*, vol. 56 (1934), p. 653.

[3] Cf., e. g., G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, p. 82. On the other hand, Hilbert has shown that it is not always possible to express a polynomial in n variables $(n \geqq 2)$ as the sum of the squares of a finite number of polynomials, "Über die Darstellung definiter Formen als Summen von Formenquadraten," *Mathematische Annalen*, vol. 32 (1888), pp. 342-350; Cf. E. Artin, "Über die Zerlegung definiter Funktionen in Quadrate," *Hamburger Abhandlungen*, vol. 5 (1926), pp. 100-115.

[4] Cf. Pólya and Szegö, *ibid.*

$$[A(x)]^2+[B(x)]^2+x\{[C(x)]^2+[D(x)]^2\}+yF(x,y),$$

where $A(x)$, $B(x)$, $C(x)$, $D(x)$ and $F(x,y)$ are polynomials with real coefficients, there will correspond to $P(x,y)$ by (1) a functional value of the form

$$\sum_{h=0}^{n}\sum_{k=0}^{n}a_ha_kd_{h+k}+\sum_{h=0}^{m}\sum_{h=0}^{m}b_hb_kd_{h+k}+\sum_{h=0}^{q}\sum_{k=0}^{q}f_hf_kd_{h+k+1}+\sum_{h=0}^{r}\sum_{k=0}^{r}g_hg_kd_{h+k+1},$$

since $c_{ij}=0$, $j\neq 0$. This expression will be non-negative if and only if every section of each of the matrices $\| d_{h+k} \|$ and $\| d_{h+k+1} \|$, $(h,k=0,1,2,\cdots)$, belongs to a non-negative definite quadratic form. This is the criterion for the solubility of the one-dimensional Stieltjes momentum problem.

3. *Case of the interval* $[-1,1]$. In view of the Legendre polynomials, it, may, perhaps, be of interest to consider also the one-dimensional momentum problem associated with the interval $[-1,1]$. Every real polynomial $P(x,y)$ which is non-negative on $(-1\leq x\leq 1;\ y=0)$ will be of the form [4]

$$[A(x)]^2+(1-x^2)\,[B(x)]^2+yF(x,y),$$

where $A(x)$, $B(x)$ and $F(x,y)$ are polynomials with real coefficients. To this will correspond, since $c_{ij}=0$, $j\neq 0$, a functional value of the form

$$\sum_{h=0}^{n}\sum_{k=0}^{n}a_ha_kd_{h+k}+\sum_{h=0}^{m}\sum_{k=0}^{m}b_hb_k(d_{h+k}-d_{h+k+2}).$$

This expression will be non-negative if and only if every section of each of the matrices $\| d_{h+k} \|$ and $\| d_{h+k}-d_{h+k+2} \|$, $(h,k=0,1,2,\cdots)$, belongs to a non-negative definite quadratic form.

4. *The trigonometrical moment problem.* Again,[4] every real trigonometric polynomial $g(\vartheta)$ of degree n which is non-negative for all values of ϑ can be represented in the form $g(\vartheta)=|h(e^{i\vartheta})|^2$, where $h(z)=a_0+a_1z+\cdots+a_nz^n$. The existence of a distribution function $\phi(E)=\Phi(\vartheta)$ satisfying (1), which now takes the form

$$c_{nm}=\int_0^{2\pi}(\cos\vartheta)^n(\sin\vartheta)^m\,d\Phi(\vartheta),$$

together with the requirement that the functional value corresponding to $|h(e^{i\vartheta})|^2$ be non-negative, requires that in the matrix

$$\| \Gamma_{i-k} \|,\ (i,k=0,1,2,\cdots),\quad\text{where}\quad \Gamma_n=\int_0^{2\pi}e^{in\vartheta}d\Phi(\vartheta),$$

every section belongs to a non-negative definite Hermitian form. This is,

in fact, the criterion for the solubility of the Herglotz trigonometric momentum problem.

5. *The one-dimensional Hausdorff momentum problem.* It is known[5] that any polynomial $f(x)$ non-negative in $[0,1]$ may be expressed as a linear combination with positive coefficients of polynomials $x^m(1-x)^{p-m}$. Then any polynomial $P(x,y)$ non-negative in $(0 \leq x \leq 1;\ y=0)$ will be of the form $f(x)+yG(x,y)$, where $G(x,y)$ is a polynomial in x and y. Consequently, the functional value P_c corresponding to such a $P(x,y)$ will be non-negative if and only if the c_{ij} are such that $c_{ij}=0$, $j \neq 0$, and

$$\sum_{k=0}^{s}(-1)^k\binom{s}{k}d_{m+k} \geqq 0, \tag{2}$$

which is the criterion for the solubility of the one-dimensional Hausdorff momentum problem.

6. *The two-dimensional Hausdorff momentum problem.*[6] Any polynomial $P(x,y)$ non-negative in $(0 \leq x \leq 1;\ 0 \leq y \leq 1)$ may be expressed similarly in terms of polynomials $x^m y^n(1-x)^{p-m}(1-y)^{q-n}$, wherefore the functional value P_c corresponding to $P(x,y)$ will be non-negative if and only if the c_{ij} satisfy a condition analogous to (2). This condition is precisely the criterion for the solubility of the two-dimensional Hausdorff momentum problem.

7. *The two-dimensional Hamburger momentum problem.* The two-dimensional Hamburger momentum problem treated in I corresponds to the case where C is the entire (x,y)-plane and hence the theorem of that paper is a particular case of our present theorem.

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[5] Pólya and Szegö, *op. cit.*, vol. 2, p. 83, ex. 49. A polynomial non-negative in $[a,b]$ can be expressed as a linear combination with positive coefficients of polynomials $(x-a)^m(b-x)^{p-m}$. In the case referred to by Pólya and Szegö, $a=-1$ and $b=1$. In the present case, $a=0$ and $b=1$.

[6] Cf. T. H. Hildebrandt and I. J. Schoenberg, "On linear functional operations and the moment problem for a finite interval in one or several dimensions," *Annals of Mathematics*, ser. 2, vol. 34 (1933), pp. 317-328; also F. Hallenbach, *Zur Theorie der Limitierungsverfahren von Doppelfolgen*, Thesis (Bonn), 1933.

SOME REMARKS ON F. JOHN'S IDENTITY.

By Hans Rademacher.

Recently F. John [1] has proved the

Theorem. *If $f(x)$ is a periodic function of bounded variation with the period 1, and if $\gamma = p/q > 1$ is a given rational number, $(p, q) = 1$, then*

$$\sum_{n=1}^{\infty} \frac{a_n(\gamma)}{n} f\left(x - \frac{\log n}{\log \gamma}\right) = \log \gamma \int_0^1 f(y)\,dy, \tag{1}$$

where $a_n(\gamma)$ is defined by

$$a_n(\gamma) = a_n(p/q) = \sum_{l=1}^{q} \exp\left[\frac{2\pi i n l}{q}\right] - \sum_{l=1}^{p} \exp\left[\frac{2\pi i n l}{p}\right]$$

or, which is the same,

$$a_n(\gamma) = \begin{cases} 0; & p \nmid n,\ q \nmid n, \\ -p; & p \mid n,\ q \nmid n, \\ q; & p \nmid n,\ q \mid n, \\ q - p; & p \mid n,\ q \mid n. \end{cases} \tag{2}$$

This interesting identity induces me to make the following three simple remarks, of which the first establishes a connection with the Riemann ζ-function, the second proves (1) for the wider realm of Riemann-integrable functions, the third gives a generalization of (1).

1. The most important special case of (1) is doubtless $f(x) = e^{2\pi i k x}$, k being an integer. If we put $\lambda^{-1} = \log \gamma$, we have to prove in this case

$$\sum_{n=1}^{\infty} \frac{a_n(\gamma)}{n} \exp[2\pi i k(x - \lambda \log n)] = \lambda^{-1} \int_0^1 e^{2\pi i k y}\,dy$$

or

$$\sum_{n=1}^{\infty} \frac{a_n(\gamma)}{n} \exp[-2\pi i k \lambda \log n] = \begin{cases} 0, & k \neq 0 \\ \lambda^{-1}, & k = 0. \end{cases} \tag{3}$$

We have

$$\sum_{n=1}^{\infty} \frac{a_n(\gamma)}{n} \exp[-2\pi i k \lambda \log n] = \sum_{n=1}^{\infty} \frac{a_n(\gamma)}{n^{1+2\pi i k \lambda}}. \tag{4}$$

But as by the definition (2) the sum $\sum_{n=1}^{N} a_n(\gamma)$ is bounded for all N, the series

$$Z(s) = \sum_{n=1}^{\infty} \frac{a_n(\gamma)}{n^s} \tag{5}$$

[1] F. John, "Identitäten zwischen dem Integral einer willkürlichen Funktion und unendlichen Reihen," *Mathematische Annalen*, vol. 110, pp. 718-721.

is convergent for $\Re(s) > 0$ and defines there a regular analytic function of s. On the other hand it follows from (2) and (5) that for $\Re(s) > 1$

$$Z(s) = \sum_{q|n} \frac{q}{n^s} - \sum_{p|n} \frac{p}{n^s} = q^{1-s} \sum_{m=1}^{\infty} \frac{1}{m^s} - p^{1-s} \sum_{m=1}^{\infty} \frac{1}{m^s}$$

or

(6) $$Z(s) = \zeta(s)(q^{1-s} - p^{1-s}).$$

The equation (6) holds for $\Re(s) > 0$. Now we have to distinguish two cases:

1) $k \neq 0$. We find from (6)

$$Z(1 + 2\pi ik\lambda) = \zeta(1 + 2\pi ik\lambda)(q^{-2\pi ik\lambda} - p^{-2\pi ik\lambda}).$$

But

$$q^{-2\pi ik\lambda} - p^{-2\pi ik\lambda} = \exp\left[-\frac{2\pi ik \log q}{\log p - \log q}\right] - \exp\left[-\frac{2\pi ik \log p}{\log p - \log q}\right] = 0,$$

since

(7) $$\frac{\log q}{\log p - \log q} = \frac{\log p}{\log p - \log q} - 1.$$

Hence

(8) $$Z(1 + 2\pi ik\lambda) = 0.$$

2) $k = 0$. In this case we have by (6)

$$Z(1) = \lim_{\epsilon \to 0} Z(1 + \epsilon) = \lim_{\epsilon \to 0} \zeta(1 + \epsilon)(q^{-\epsilon} - p^{-\epsilon}) = \lim_{\epsilon \to 0} \frac{q^{-\epsilon} - p^{-\epsilon}}{\epsilon},$$

(9) $$Z(1) = -\log q + \log p = \log \gamma = \lambda^{-1}.$$

The formulae (4), (5), (8), (9) prove (3).

By means of a Fourier expansion, the equation (3) could, of course, be used to prove (1) for a rather extended class of functions $f(x)$. But this reasoning would involve some complications of convergence, which can be surmounted easily only for functions with absolutely convergent Fourier-series, e. g., functions with bounded derivative. However, instead of pursuing this method, we proceed to prove (1) directly in our next remark.

2. In order to study the expression

$$\lim_{N \to \infty} \sum_{n=1}^{\infty} \frac{a_n(\gamma)}{n} f(x - \lambda \log n)$$

for a Riemann-integrable function $f(y)$ it is obviously sufficient to consider only such N as are divisible by pq, since the $a_n(\gamma)$ and $f(y)$ are bounded. Now we have by (2)

$$
\begin{aligned}
S_M(f) &= \sum_{n=1}^{Mpq} \frac{a_n(\gamma)}{n} f(x-\lambda \log n) \\
&= \sum_{\substack{1\leqq n \leqq Mpq \\ q|n}} \frac{q}{n} f(x-\lambda\log n) - \sum_{\substack{1\leqq n\leqq Mpq \\ p|n}} \frac{p}{n} f(x-\lambda \log n) \\
&= \sum_{m=1}^{Mp} \frac{1}{m} f(x-\lambda\log mq) - \sum_{m=1}^{Mq} \frac{1}{m} f(x-\lambda \log mp) \\
&= \sum_{m=Mq+1}^{Mp} \frac{1}{m} f(x-\lambda\log m - \lambda \log q),
\end{aligned}
$$

upon making use of (7) and of the periodicity of $f(y)$. Moreover, there is no loss of generality in setting $x=\lambda \log q$, for if (1) is true for any special value x_0, it is also true for any other x, as $f(y)$ and $f(y-x_0+x)$, regarded as functions of y, are both periodic and Riemann-integrable. Hence all we have to prove is

$$
\lim_{M\to\infty} S_M(f) = \lim_{M\to\infty} \sum_{m=Mq+1}^{Mp} \frac{1}{m} f(-\lambda \log m) = \lambda^{-1} \int_0^1 f(y)\,dy. \tag{10}
$$

Now let us first treat the special "step-function"

$$
\phi_\alpha(y) = \begin{cases} 1, & 0 \leqq y < \alpha \\ 0, & \alpha \leqq y < 1, \end{cases} \tag{11}
$$

$\phi_\alpha(y)$ being defined in points outside the interval $0 \leqq y < 1$ by periodic repetition of (11) modulo 1. The parameter α is supposed to be such that $0 \leqq \alpha \leqq 1$; for the extreme values $\alpha = 0$ or 1 one of the two inequalities in (11) cannot be fulfilled. We have therefore $\phi_0(y)=0$ and $\phi_1(y)=1$ for all y.

For $f=\phi_\alpha$ (10) becomes

$$
\lim_{M\to\infty} S_M(\phi_\alpha) = \lim_{M\to\infty} \sum_{m=Mq+1}^{Mp} \frac{1}{m} \phi_\alpha(-\lambda\log m) = \alpha\lambda^{-1}. \tag{12}
$$

As this is trivially true for $\alpha=0$, we can assume $0<\alpha\leqq 1$. Now we have by (11)

$$
\sum_{m=Mq+1}^{Mp} \frac{1}{m} \phi_\alpha(-\lambda \log m) = \sum_{\substack{Mq<m\leqq Mp \\ 0 \leqq -\lambda \log m < \alpha \ (\mathrm{mod}\ 1)}} \frac{1}{m}, \tag{13}
$$

where $0 \leqq x < \alpha \pmod 1$ means, of course, $0 \leqq x - [x] < \alpha$. But the conditions of summation on the right-hand side of (13) can be written

$$
\lambda(\log M + \log q) < \lambda \log m \leqq \lambda(\log M + \log p), \tag{a}
$$

$$
1-\alpha < \lambda\log m \leqq 1 \pmod 1. \tag{b}
$$

Condition (a) assigns to $\lambda \log m$ an interval of length $\lambda(\log p - \log q) = 1$. Thus of the infinite set of intervals (b) of length α, which are periodic

modulo 1, either just one falls in (a) or two parts of intervals of (b), together of length α, lie in (a), so that the summation in (13) is either of the type

$$\lambda(\log M + \log q) + u_M < \lambda \log m \leqq \lambda(\log M + \log q) + u_M + \alpha,$$

where $0 \leqq u_M \leqq 1 - \alpha$, or of the type

$$\begin{aligned} &\lambda(\log M + \log q) < \lambda \log m \leqq \lambda(\log M + \log q) + \beta_1 \\ &\lambda(\log M + \log p) - \beta_2 < \lambda \log m \leqq \lambda(\log M + \log p), \qquad \beta_1 + \beta_2 = \alpha. \end{aligned}$$

Therefore (12) will be proved if we show that

$$\lim_{M\to\infty} \sum_{\log M + v < \log m \leqq \log M + v + \beta\lambda^{-1}} \frac{1}{m} = \beta\lambda^{-1}, \tag{14}$$

where $v = v_M$ may be any number lying between assigned bounds, $c \leqq v_M < C$. Now for $m > 1$

$$\int_m^{m+1} \frac{dt}{t} < \frac{1}{m} < \int_{m-1}^{m} \frac{dt}{t}$$

and consequently

$$\int_{Me^v}^{Me^{v\gamma^{\beta}}} \frac{dt}{t} < \sum_{Me^v < m \leqq Me^{v\gamma^{\beta}}} \frac{1}{m} < \frac{1}{Me^v} + \int_{Me^v}^{Me^{v\gamma^{\beta}}} \frac{dt}{t}$$

or

$$\beta < \sum_{Me^v < m \leqq Me^{v\gamma^{\beta}}} \frac{1}{m} < \frac{1}{Me^c} + \beta,$$

which proves (14) and therefore also (12).

Now any periodic step-function of period 1 can be built up as a linear combination of a finite number of step-functions $\phi_a(y)$ of the special type (11) with different parameters α. Hence (10) is proved for arbitrary step-functions with a finite number of steps.

If, finally, $f(y)$ is a periodic Riemann-integrable function, we can, to any given $\epsilon > 0$, assign two step-functions $\phi(y)$ and $\Phi(y)$ of period 1, such that

$$\phi(y) \leqq f(y) \leqq \Phi(y) \tag{15}$$

and

$$\int_0^1 (\Phi(y) - \phi(y))\,dy < \epsilon. \tag{16}$$

Since (10) is valid for $\phi(y)$ and $\Phi(y)$, we have

$$\begin{aligned} &\lim_{M\to\infty} S_M(\phi) = \lambda^{-1} \int_0^1 \phi(y)\,dy, \\ &\lim_{M\to\infty} S_M(\Phi) = \lambda^{-1} \int_0^1 \Phi(y)\,dy. \end{aligned} \tag{17}$$

But $S_M(f)$ shows in (10) only *positive* coefficients of $f(-\lambda \log m)$ and therefore we have from (15)

$$S_M(\phi) \leqq S_M(f) \leqq S_M(\Phi).$$

From this and (17) we conclude

$$\lambda^{-1}\int_0^1 \phi(y)\,dy \leqq \varliminf_{M\to\infty} S_M(f) \leqq \varlimsup_{M\to\infty} S_M(f) \leqq \lambda^{-1}\int_0^1 \Phi(y)\,dy.$$

But according to (16) this proves (10) for any Riemann-integrable function $f(y)$, for which, therefore, John's identity (1) is true.

3. The relation between the identity (1) and the Riemann ζ-function, discussed in § 1, suggests the possibility of finding similar identities related to other ζ-functions.

Let K be a field of algebraic numbers, of degree n; let γ be a number of the field with $|N(\gamma)| > 1$ and

$$\gamma = \mathfrak{a}/\mathfrak{b}, \qquad (\mathfrak{a}, \mathfrak{b}) = 1.$$

Now for ideals $\mathfrak{n}$ of the field we introduce, in analogy with (2), the arithmetic function $a_{\mathfrak{n}}(\gamma)$ through the definition

$$a_{\mathfrak{n}}(\gamma) = \begin{cases} 0 & \mathfrak{a}\nmid\mathfrak{n} \quad \mathfrak{b}\nmid\mathfrak{n} \\ -N(\mathfrak{a}) & \mathfrak{a}\mid\mathfrak{n} \quad \mathfrak{b}\nmid\mathfrak{n} \\ N(\mathfrak{b}) & \mathfrak{a}\nmid\mathfrak{n} \quad \mathfrak{b}\mid\mathfrak{n} \\ N(\mathfrak{b}) - N(\mathfrak{a}) & \mathfrak{a}\mid\mathfrak{n} \quad \mathfrak{b}\mid\mathfrak{n}. \end{cases} \tag{18}$$

Let $\mathfrak{C}$ be a class of ideals of K. We shall then prove the

THEOREM. *If $f(x)$ is R-integrable and of period 1, the equation*

$$\sum_{\mathfrak{n}\in\mathfrak{C}} \frac{a_{\mathfrak{n}}(\gamma)}{N(\mathfrak{n})} f\left(x - \frac{\log N(\mathfrak{n})}{\log|N(\gamma)|}\right) = \kappa \log|N(\gamma)| \int_0^1 f(y)\,dy \tag{19}$$

is valid, the summands on the left-hand side being arranged according to increasing $N(\mathfrak{n})$. In (19) κ is a constant depending only on the field, namely

$$\kappa = \frac{2^{r_1+r_2}\pi^{r_2}R}{w\,|\sqrt{d}\,|}$$

(r_1 real and $2r_2$ complex fields among the conjugate fields, w number of roots of unity contained in K, *d discriminant, R regulator of* K).

We could repeat our argument of § 1 with one change: viz., the con-

vergence of the series

$$Z(s)=\sum_{\mathfrak{n}\in\mathfrak{C}}\frac{a_{\mathfrak{n}}(\gamma)}{N(\mathfrak{n})^{s}}$$

can be proved only for $\Re(s)>1-[2/(n+1)]$, for which purpose we should have to use Landau's estimate [2] of the "ideal-function"

$$H(x;\mathfrak{C})=\sum_{\substack{\mathfrak{n}\in\mathfrak{C}\\ N(\mathfrak{n})\leqq x}}1=\kappa\log x+O(x^{1-[2/(n+1)]}). \tag{20}$$

But instead of giving further details of this reasoning we prefer to pass immediately to the generalization of § 2, which is not quite so obvious.

For our proof we start with the remark that for a fixed A

$$\sum_{\substack{\mathfrak{n}\in\mathfrak{C}\\ MA<N(\mathfrak{n})\leqq(M+1)A}}\frac{1}{N(\mathfrak{n})}\to 0 \quad\text{as}\quad M\to\infty.$$

Indeed, we have from (20)

$$\sum_{\substack{\mathfrak{n}\in\mathfrak{C}\\ MA<N(\mathfrak{n})\leqq(M+1)A}}\frac{1}{N(\mathfrak{n})}\leqq\frac{1}{MA}\sum_{\substack{\mathfrak{n}\in\mathfrak{C}\\ MA<N(\mathfrak{n})\leqq(M+1)A}}1$$

$$=\frac{1}{MA}\kappa A+O\left(\frac{1}{M}M^{1-[2/(n+1)]}\right)=O(M^{-[2/(n+1)]}).$$

Hence for the study of

$$\lim_{N\to\infty}\sum_{\substack{\mathfrak{n}\in\mathfrak{C}\\ N(\mathfrak{n})\leqq N}}\frac{a_{\mathfrak{n}}(\gamma)}{N(\mathfrak{n})}f(x-\Lambda\log N(\mathfrak{n})),$$

where $\Lambda=(\log|N(\gamma)|)^{-1}$, it is sufficient to consider only such N as are divisible by $N(\mathfrak{a}\mathfrak{b})$, and to determine the limit of

$$S^*_M(f)=\sum_{\substack{\mathfrak{n}\in\mathfrak{C}\\ N(\mathfrak{n})\leqq MN(\mathfrak{a}\mathfrak{b})}}\frac{a_{\mathfrak{n}}(\gamma)}{N(\mathfrak{n})}f(x-\Lambda\log N(\mathfrak{n}))$$

as $M\to\infty$. Now we have by (18)

$$S^*_M(f)=\sum_{\substack{\mathfrak{n}\in\mathfrak{C}\\ \mathfrak{b}\mid\mathfrak{n}\\ N(\mathfrak{n})\leqq MN(\mathfrak{a}\mathfrak{b})}}\frac{N(\mathfrak{b})}{N(\mathfrak{n})}f(x-\Lambda\log N(\mathfrak{n}))-\sum_{\substack{\mathfrak{n}\in\mathfrak{C}\\ \mathfrak{a}\mid\mathfrak{n}\\ N(\mathfrak{n})\leqq MN(\mathfrak{a}\mathfrak{b})}}\frac{N(\mathfrak{a})}{N(\mathfrak{n})}f(x-\Lambda\log N(\mathfrak{n})).$$

The conditions $\mathfrak{b}\mid\mathfrak{n}$ and $\mathfrak{a}\mid\mathfrak{n}$ may be replaced by $\mathfrak{n}=\mathfrak{b}\mathfrak{m}_1$, $\mathfrak{n}=\mathfrak{a}\mathfrak{m}_2$ respectively. As we have $\mathfrak{a}=\gamma\mathfrak{b}$, $\mathfrak{a}$ and $\mathfrak{b}$ belong to the same class, say $\mathfrak{C}_1$. Therefore the ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$ lie both in the class

[2] E. Landau, *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und Ideale*, Leipzig, 1918, p. 131, Satz 210.

$$\mathfrak{C}' = \mathfrak{C}\mathfrak{C}_1^{-1}.$$

Hence we get, writing $\mathfrak{m}$ instead of $\mathfrak{m}_1$ and $\mathfrak{m}_2$,

$$S^*_M(f) = \sum_{\substack{\mathfrak{m}\in\mathfrak{C}' \\ N(\mathfrak{m})\leqq MN(\mathfrak{a})}} \frac{1}{N(\mathfrak{m})} f(x - \Lambda(\log N(\mathfrak{m}) + \log N(\mathfrak{b})))$$
$$- \sum_{\substack{\mathfrak{m}\in\mathfrak{C}' \\ N(\mathfrak{m})\leqq MN(\mathfrak{b})}} \frac{1}{N(\mathfrak{m})} f(x - \Lambda(\log N(\mathfrak{m}) + \log N(\mathfrak{a}))).$$

Because of $\Lambda^{-1} = \log |N(\gamma)| = \log N(\mathfrak{a}) - \log N(\mathfrak{b})$, we have

$$\Lambda(\log N(\mathfrak{m}) + \log N(\mathfrak{b})) - \Lambda(\log N(\mathfrak{m}) + \log N(\mathfrak{a})) = -1.$$

From this and the periodicity of $f(y)$ we conclude

$$S^*_M(f) = \sum_{\substack{\mathfrak{m}\in\mathfrak{C}' \\ MN(\mathfrak{b}) < N(\mathfrak{m})\leqq MN(\mathfrak{a})}} \frac{1}{N(\mathfrak{m})} f(x - \Lambda\log N(\mathfrak{m}) - \Lambda\log N(\mathfrak{b})).$$

As we saw in § 2, the choice of a special value for x involves no loss of generality. We put $x = \Lambda \log N(\mathfrak{b})$ and then have to prove

$$(21)\qquad \lim_{M\to\infty} S^*(f) = \lim_{M\to\infty} \sum_{\substack{\mathfrak{m}\in\mathfrak{C}' \\ MN(\mathfrak{b}) < N(\mathfrak{m})\leqq MN(\mathfrak{a})}} \frac{1}{N(\mathfrak{m})} f(-\Lambda\log N(\mathfrak{m})) = \Lambda^{-1}\int_0^1 f(y)\,dy.$$

For the required proof we need the relation

$$(22)\qquad L(x) = \sum_{\substack{\mathfrak{m}\in\mathfrak{C}' \\ N(\mathfrak{m})\leqq x}} \frac{1}{N(\mathfrak{m})} = \kappa\log x + C + O(x^{-[2/(n+1)]}),$$

which follows from (20) by the customary process of Abel's partial summation. The constant C in (22) may depend on the class $\mathfrak{C}'$.

In complete analogy with § 2, we prove (21) only for the special step-function $\phi_a(y)$, defined in (11). We have

$$S^*_M(\phi_a) = \sum_{\substack{\mathfrak{m}\in\mathfrak{C}' \\ MN(\mathfrak{b}) < N(\mathfrak{m})\leqq MN(\mathfrak{a})}} \frac{1}{N(\mathfrak{m})} \phi_a(-\Lambda\log N(\mathfrak{m})) = \sum_{\substack{\mathfrak{m}\in\mathfrak{C}' \\ MN(\mathfrak{b}) < N(\mathfrak{m})\leqq MN(\mathfrak{a}) \\ 0\leqq -\Lambda\log N(\mathfrak{m}) < a \pmod 1}} \frac{1}{N(\mathfrak{m})}.$$

The conditions of this sum may be treated like those of (13), and our problem is then reduced to the proof of

$$\lim_{M\to\infty} \sum_{\substack{\mathfrak{m}\in\mathfrak{C}' \\ \log M + v < \log N(\mathfrak{m}) \leqq \log M + v + \beta\Lambda^{-1}}} \frac{1}{N(\mathfrak{m})} = \kappa\beta\Lambda^{-1}.$$

According to (22) the left-hand side is equal to

$$\begin{aligned}&\lim_{M\to\infty}\{L(Me^v \mid N(\gamma)|^\beta) - L(Me^v)\}\\ &=\lim_{M\to\infty}\{\kappa(\log M + v + \beta\Lambda^{-1}) - \kappa(\log M + v) + O(M^{-[2/(n+1)]})\}\\ &= \kappa\beta\Lambda^{-1},\end{aligned}$$

which was to be proved.

The further arguments are quite the same as in § 2. We first consider arbitrary step-functions and can then enclose a given R-integrable function $f(y)$ between two step-functions $\phi(y)$ and $\Phi(y)$ whose integrals differ by as little as we wish. In this way the theorem of this paragraph is fully proved.

I close this article with a special example of the generalized John's identity (19). Let K be Gauss's field of complex numbers $a + bi$. We choose $\gamma = 1 + i$. As there is only the principal class of ideals, we can replace the ideals by integers $m + ni$ of the field. We have only to observe that each principal ideal is represented by four associated numbers. If we therefore sum over all integers (with the omission of 0), we get on both sides of (19) the four-fold amount. We notice further that $\kappa = \pi/4$ in this case and that

$$a_{\mathfrak{n}}(\gamma) = a_{m+ni}(1+i) = \begin{cases} 1, & m \not\equiv n \pmod 2\\ -1, & m \equiv n \pmod 2,\end{cases}$$

or

$$a_{m+ni}(1+i) = (-1)^{m+n+1}.$$

Hence we have the equation

$$\sum_{m,n}{}' \frac{(-1)^{m+n+1}}{m^2+n^2} f\left(x - \frac{\log(m^2+n^2)}{\log 2}\right) = \pi \log 2 \int_0^1 f(y)\,dy,$$

the sum being extended over all pairs (m, n) with the omission of $(0, 0)$ and arranged according to increasing values of $m^2 + n^2$.

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NOTE ON A THEOREM OF PONTRJAGIN.

By E. R. van Kampen.

I. The problem to determine under what conditions a locally euclidean group is a Lie group was solved for the compact case by v. Neumann [1] [1]. Later Pontrjagin [2] generalized his solution by proving the following theorem:

(1) *Any locally connected finite dimensional compact group is a Lie group.*[2]

In this note [3] we prove certain theorems related to (1) and implying (1). Some of these have already been stated in [5], section IX. The proofs follow easily from the structural considerations in [6] and similar more detailed results on abelian groups to be found in [3] and also in [4]. We wish to emphasize the central position occupied by Theorem 6 ([5], section IX, restated in [6]). This theorem, obtained as a corollary from the theory of almost periodic functions, can then be used as sole foundation of the structural theory of compact groups. While [5] has not been written with this idea in mind, an analysis of the line of thought used there will justify this statement for the case of abelian compact groups as well.

While we are considering compact separable groups, there is no difficulty at all in extending the results to bicompact groups.

II. *The closed invariant subgroups of a compact connected group.* By the same methods as used in [6], II-IV, we prove first:

(2) *Any closed invariant subgroup G of a compact connected group F can be generated by a closed subgroup of the centrum of F and certain of the invariant simple subgroups $S^{(l)}$ of F.*

Each compact Lie group F_n ([6], p. 301) contains a closed invariant subgroup G_n corresponding to G. For these groups (2) is well known. If for a certain number n and $l < p_n$ ([6], p. 302) the subgroup $S_n^{(l)}$ of F_n corresponding to $S^{(l)}$ is contained in G_n, then the same is true for all $m > n$, and $S^{(l)}$ is contained in G. Moreover G_n is generated by its centrum C'_n and

[1] The numbers in square brackets refer to the literature at the end of the note.

[2] In order to simplify the wording of certain statements, we allow Lie groups to have a finite number of components. In particular we include under that term as a degenerate case all finite groups.

[3] It can be considered as a continuation of [6] from which we take over all notations, in particular those of Theorems 1 and 2.

the subgroups $S_n^{(l)}$ contained in G_n. Here C'_n is the common part of G_n and the centrum C_n of F_n. Hence G is generated by its subgroups $S^{(l)}$ together with the common part of G and C.

III. *The dimension of a compact group.*

(3) *The dimension of a compact group F is equal to the sum of the dimensions of its simple Lie subgroups $S^{(l)}$ $(l = 1, 2, \cdots)$, and the dimension of its centrum C.*

We may of course suppose that F is connected and that C has the finite dimension m. If the number of groups $S^{(l)}$ is not finite, we can find locally euclidean sets of arbitrarily high dimension by considering the groups generated by certain finite collections of groups $S^{(l)}$. Thus we can assume that the number of groups $S^{(l)}$ is finite, and that they generate a Lie group S of dimension n. Considering that in this case the group A ([6], Theorem 1) is finite, so that F and F/A are locally homeomorphic, and applying the reasoning of [4], p. 458, h, on the centrum C, we find that a nucleus of F is homeomorphic with the product of an n-cell, an m-cell and a 0-dimensional set. Hence statement (3) is proved.

(4) *If H is a closed invariant subgroup of a compact group F, then the dimension of F is equal to the sum of the dimensions of H and F/H.*

We may obviously suppose that F is connected. Comparing (2) with (3) we see that (4) immediately reduces to the case of compact abelian groups. But in this case statement (4) follows from [4], p. 458, h.

(5) *Any sufficiently small subgroup H of a finite dimensional compact group F is 0-dimensional.*

We may again suppose that F is connected. Then F has a 0-dimensional invariant subgroup G with a Lie factorgroup. According to (3) and [6], Theorem 1, this is true for F, if it is true for the centrum C of F. But for C it is an immediate consequence of the properties of charactergroups (Compare [4]).

As a Lie group does not have arbitrarily small subgroups, we can restrict H to such an open set containing G that the image of H in F/G consists of the identity element only. But then H is contained in G, so that it is 0-dimensional.

The most noteworthy point about the proofs of (4) and (5) is that a thorough analysis of the structure of compact groups on the lines indicated

is at present unavoidable. As a result these simple sounding theorems cannot yet be proved for any class of groups for which no structural analysis is known. Important examples are the class of locally compact groups and the very restricted class of locally euclidean groups.

IV. In this section we consider a closed invariant subgroup G of a compact connected group F from the point of view of local connectedness. According to (2) the group G has subgroups S', C', A', related in the same way as the subgroups S, C, A of F respectively ([6], Theorem 1). The subgroup S' is generated by the simple Lie subgroups invariant in G, C' is the centrum of G and A' is the common part of S' and C'. According to [6], Theorem 3, the groups F and C/A are at the same time locally connected and not locally connected. Though G is not connected, Theorem 3 remains true for G and C'/A'. For C'/A' and G/S are simply isomorphic and S is connected. Hence there is a one-to-one correspondence between the components of G and the components of C'/A'.

By [6], Theorem 3, the components of the identity elements of G and C'/A' are locally connected and not locally connected at the same time. As we have just seen that G and C'/A' have at the same time a finite or an infinite number of components, it follows now that G and C'/A' are at the same time locally connected or not locally connected.

We consider the case that both F and C/A are locally connected, while G and C'/A' are not locally connected.

In the first place it is then possible that the factorgroup $A''' = A''/A'$ of A' in the common part A'' of C' and A is not locally connected. As A is 0-dimensional this means simply that A''' is not finite. Then A/A' is not finite, S/S' must be infinite dimensional and F/G must be infinite dimensional also.

In the second place we suppose that the factorgroup A''' is locally connected and accordingly finite. Then the factorgroup $C'' = C'/A''$ of the common part A'' of A and C' in C' is locally simply isomorphic with the factorgroup C'/A', so $C'' = C'/A''$ is not locally connected. The group C'' can be considered as a subgroup of the locally connected group C/A. Now a locally connected compact abelian group is the direct product of a certain collection of rotation groups. If such a group has a non-locally connected closed subgroup, the factorgroup cannot be finite dimensional. So we see:

(6) *If a closed invariant subgroup G of a locally connected compact group F is itself not locally connected then F/G is not finite dimensional.*

Pontrjagin's theorem (cited in (1)) is an immediate consequence of (5), (6) and the existence of arbitrarily small invariant subgroups with Lie factor-groups in any compact group. However, once (3) is proved (1) follows more easily by a direct argument:

In a finite dimensional compact group F the number of subgroups $S^{(1)}$ is finite, so that A is finite ([6], Theorem 1). But then, if F is also locally connected, C is locally connected ([6], Theorem 3) and as C is also finite dimensional it is a Lie group. Hence F is a Lie group.

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[5] E. R. van Kampen, "Almost periodic functions and compact groups," *Annals of Mathematics*, vol. 37 (1936).

[6] E. R. van Kampen, "The structure of a compact connected group," *American Journal of Mathematics*, vol. 57 (1935), pp. 301-308.

POINT SET THEORY APPLIED TO THE RANDOM SELECTION OF THE DIGITS OF AN ADMISSIBLE NUMBER.[1]

By Arthur H. Copeland.

Kamke and Tornier have pointed out objections to von Mises' definition of the collective (*Kollektiv*). The difficulty arises in connection with a selection operator (*Auswahl*) whose function is to transform one collective into another. In this paper we shall show how the difficulty can be overcome.

The collective is the fundamental element in von Mises' theory of probability.[2] It is the mathematical idealization of a sequence of physical occurrences (measurements, results of tossing a coin, etc.). A collective K consists of an infinite sequence of elements (points of some space S). Thus

$$(1) \qquad K = e^{(1)}, e^{(2)}, e^{(3)}, \cdots$$

The points of S are called labels (*Merkmale*) and the space S is called the label space (*Merkmalraum*). Associated with every label is a probability defined as follows. Let m be any label belonging to S and let r_n be the number of times the label m occurs in the first n terms of K. Then r_n/n is called the success ratio for m in the first n trials of K and $\lim_{n\to\infty} r_n/n$ is called the probability of m with respect to K. The set of probabilities associated with the elements of S is called the distribution (*Teilung*). The first restriction imposed on the collective is that it must possess a distribution, i. e., the limit of the success ratio must exist for every element m of S.

The operation of "selection" is defined in the following manner. Let

$$(2) \qquad n_1, n_2, n_3, \cdots$$

be any infinite increasing sequence of positive integers. We can form a new collective K' by selecting the n_1-st, n_2-nd, n_3-rd, $\cdots$ terms from $K = e^{(1)}, e^{(2)}, e^{(3)}, \cdots$. Thus $K' = e^{(n_1)}, e^{(n_2)}, e^{(n_3)}, \cdots$. It will be convenient to introduce a notation for this operation. Let us define the sequence

$$(3) \qquad x = x^{(1)}, x^{(2)}, x^{(3)}, \cdots$$

[1] Presented to the Society April 11, 1930. This paper also contains the material presented Dec. 29, 1928, under the title *A proof that almost every number is admissible and is associated with the probability one-half.*

[2] See von Mises I and II. References to literature are given at the end of this paper.

such that the terms $x^{(n_1)}, x^{(n_2)}, x^{(n_3)}, \cdots$ are all 1's and the rest of the terms are all 0's. Then x is a sequence similar to K. The corresponding label space consists of the two points 1 and 0. It is also convenient to think of x as a number which lies between 0 and 1, is expressed in the binary scale, and has the digits $x^{(1)}, x^{(2)}, x^{(3)}, \cdots$. Any number x in the interval from 0 to 1 (0 excluded) can be a selection operator. In the case of the *ambiguous* representation, the ambiguity is decided by the condition that the sequence (2) is infinite and hence an infinite number of the digits of x must be equal to 1. The fact that K' is obtained by operating on K with x is expressed by the equation $K' = K \subset x$.[3]

The second restriction on the collective is that its distribution must be invariant under all selections which operate "*Ohne Benützung der Merkmalunterschiede.*" Roughly this means that the distribution of a collective K is invariant under the operation of any selection which is random with respect to K. Von Mises regards as random any selection which is given by mathematical law independently of the collective. Kamke raises the following objection to this restriction. "*Es fragt sich, ob diese Genügsamkeit angebracht ist. Denn unterliegen die Folgen* (8)"[4] "*keinerlei Einschränkung, so kann man unabhängig von jeder E-Folge*"[4] (sequence (1)) "*die Gesamtheit aller folgen* (8) *bilden und für jede W-Folge*[4] *gäbe es dann unter diesen, unabhängig von der W-Folge gebildeten Index-folgen* (8), *stets eine solche, bei der auf jedes* E_ν *das Merkmal m zutrifft, wie auch eine solche, bei der auf kein* E_ν *das Merkmal m zutrifft. Dann gäbe es also überhaupt keine Kollektiv. Man darf also offenbar nicht beliebige Folgen* (8) *zulassen, sondern muß sich auf 'gesetzmäßige' oder 'mathematisch gegebene' Folgen* (8) *beschränken. Wie der Bereich dieser Folgen gegen die 'nicht gesetzmäßigen' Folgen abzugrenzen ist, diese Frage bleibt aber offen.*[5]

For the purpose of this paper it will be convenient to state Kamke's criticism in a slightly different form. To this end we shall replace the second

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[3] The symbol $\subset$ is an inverted implication sign. If we think of K and x as both representing event sequences, then $K \subset x$ represents the event sequence "K if x," or "K is implied by x." (See Copeland III.) Dörge indicates the operation of selection by a product. (See Dörge I.) Since I have used the product $x \cdot y$ to indicate the event sequence x and y, it is necessary to use another symbol to indicate the event sequence "x if y." It is interesting to note that Dörge's *Einheit Auswahl* is represented in my notation by the number 1.

[4] Sequence (8) referred to by Kamke is the same as sequence (2) in my paper. An "E-Folge" is an arbitrary sequence K. An E-Folge is called a "W-Folge" with respect to a label m, provided, there exists a probability of m with respect to the sequence.

[5] Kamke I. Tornier points out a similar objection, Tornier I.

restriction on the collective by the following. The distribution of a collective must be invariant under the operation of every selection of a certain set E. We shall call the set E a *fundamental set* and for the present we shall leave it entirely undefined. Kamke points out that if the fundamental set consists of all selections which can be defined by mathematical law, then the set of collectives will be null.

Corresponding to any fundamental set, there arises the question of consistency of the two restrictions on the collective. We shall establish consistency by proving the existence of sequences satisfying the restrictions. It will be sufficient to consider a restricted type of sequence. We shall choose a label space which consists of the numbers 1 and 0. The sequences associated with this space admit of the same numerical representation as the selection operators. The reason for choosing such a restricted label space is that von Mises imposes certain other conditions with which this paper is not concerned but which are satisfied vacuously by the sequences associated with this label space. The space represents a simple alternative situation (heads or tails, etc.). We shall let the label 1 represent a success and the label 0 a failure. With this situation we can obtain the following simple expression for the success ratio.

$$p_n(K) = \sum_{i=1}^{n} (e^{(i)}/n) \tag{4}$$

where $p_n(K)$ is the success ratio of the label 1 in the first n trials of K. We shall denote the probability by $p(K)$ where $p(K) = \lim_{n\to\infty} p_n(K)$.

As an example of a fundamental set, we may take the set consisting of all selections whose numerical representations have the form

$$x_{r,n} = 2^{-r}/(1 - 2^{-n}) = 2^{-r} + 2^{-r-n} + 2^{-r-2n} - \cdots,$$

where r and n are integers and $0 < r \leqq n$. Then the terms $x^r, x^{r+n}, x^{r+2n}, \cdots$ are all 1's, and the rest of the terms are all 0's. Hence, the operator $x_{r,n}$ selects the r-th, the $(r+n)$-th, the $(r+2n)$-th, $\cdots$ terms of K. It has been proved that there exists a set of sequences whose distributions are invariant under the operation of all selections of this fundamental set.[6] For these sequences, the associated label space consists of the elements 1 and 0. These sequences are called admissible numbers.[6] Admissible numbers also display the characteristics of the Bernoulli series, that is the probability of r successes in n trials is ${}_nC_r\, p^r(K)[1 - p(K)]^{n-r}$. In order that they may do

[6] For the proof of the existence of admissible numbers see Copeland I. A much simpler and more elegant proof has recently been given by von Mises. See von Mises III, example VI.

this, they must satisfy a certain independence condition which is defined in the following manner. Given two sequences K_1 and K_2, we can form a third sequence $K_1 \cdot K_2$ whose i-th term is a success (i. e., a 1) if and only if the i-th terms of both K_1 and K_2 are successes. Thus the i-th term of $K_1 \cdot K_2$ is the algebraic product of $e_1^{(i)}$ and $e_2^{(i)}$, and

$$(5) \qquad K_1 \cdot K_2 = e_1^{(1)} \cdot e_2^{(1)}, e_1^{(2)} \cdot e_2^{(2)}, \; e_1^{(3)} \cdot e_2^{(3)}, \cdots$$

The sequences K_1 and K_2 are independent if and only if

$$p(K_1 \cdot K_2) = p(K_1) \cdot p(K_2).$$

The conjunction $K_1 \cdot K_2 \cdots K_n$ of the sequences $K_1, K_2, \cdots K_n$ is defined in the same manner as that of the conjunction of two sequences. A necessary and sufficient condition that n sequences $K_1, K_2, \cdots K_n$ be independent is that, for every subset $K_{r_1}, K_{r_2}, \ldots K_{r_\nu}$ of the set $K_1, K_2, \cdots K_n$,

$$p(K_{r_1} \cdot K_{r_2} \cdots K_{r_\nu}) = p(K_{r_1}) \cdot p(K_{r_2}) \cdots p(K_{r_\nu}).$$

A necessary and sufficient condition that a sequence K be an admissible number is that there exists a number $p(0 < p < 1)$ such that for every set of integers $r_1, r_2, \cdots r_\nu, n$ where $0 < r_1 < r_2 < \cdots < r_\nu \leqq n$,

$$(6) \qquad p[(K \subset x_{r_1,n}) \cdot (K \subset x_{r_2,n}) \cdots (K \subset x_{r_\nu,n})] = p^\nu.$$

If K is an admissible number, then in particular $p(K \subset x_{r,n}) = p$. Furthermore, if r and n are both 1, then $x_{r,n}$ is the identity operator and $p(K) = p$. Thus K possesses a distribution and this distribution is invariant under all selections of the form $x_{r,n}$.[7] Moreover, the sequences

$$K \subset x_{1,n}, \; K \subset x_{2,n}, \cdots K \subset x_{n,n}$$

are independent.

Certain other properties of sequences can be invariant under the operation of selection. For example, it may happen that not only the distribution of a sequence is invariant under selection, but its property of being a collective is also invariant. Let $A(p)$ denote the set of all admissible numbers associated with the probability p. We shall say that *the properties of an admissible number K are invariant under the operation of a selection x, if $K \subset x$ belongs to the same set $A(p)$ as K.* I have proved that properties of all admissible numbers are invariant under all selections of the form $x_{r,n}$.

It will be observed that an admissible number is a collective whose fundamental set consists of the operators $x_{r,n}$. Thus, we already have one way of

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[7] In my previous papers I have used the notation $(r/n)K$ instead of $K \subset x_{r,n}$.

getting around the difficulty raised by Kamke and Tornier. However, it might well be obejcted that this fundamental set is too restricted. For this reason we shall consider to what extent this set can be altered or augmented. At the conclusion of this paper we shall discuss other contributions to this problem.

We shall show that, corresponding to any denumerable fundamental set, there exists a continuum of admissible numbers whose properties are invariant under the operation of all selections of the fundamental set. A fundamental set can consist of almost every number from 0 to 1, "almost every" being used in the Lebesgue sense. The corresponding set of admissible numbers will then be at least denumerable. On the basis of the assumption of a well ordered continuum, we shall prove that both the fundamental set and the corresponding set of admissible numbers can be non-denumerable.

It seems to me sufficient to choose a denumerable fundamental set for the following reason. Let D be any denumerable fundamental set and let M be a set of admissible numbers. We shall show that M can contain almost every number in the interval from 0 to 1 and that if K is any member of M, then the properties of K will be invariant under the operation of every selection of D. Moreover, corresponding to any element K of M, there will exist a set E_K which contains almost every selection and all selections of which leave invariant the properties of K. It is important to notice that E_{K_1} and E_{K_2} are not necessarily identical if K_1 and K_2 are distinct.

The following theorem relates to the choice of the fundamental set.

THEOREM 1. *Given any denumerable set of selections D, there exists a set of admissible numbers M which has the power of the continuum and which is such that the operation of any selection of D on any admissible number of M, leaves invariant the properties of that admissible number.*

Let x be an arbitrary selection. Then $x = x^{(1)}, x^{(2)}, x^{(3)}, \cdots$ where $x^{(i)} = 1$ or 0. We shall let the set M consist of admissible numbers associated with the probability b/a, where a and b are integers such that $0 < b < a$. We shall let $y = y^{(1)}, y^{(2)}, y^{(3)}, \cdots$ where $y^{(i)} = 0, 1, 2, \cdots (a-1)$ and $K = e^{(1)}, e^{(2)}, e^{(3)}, \cdots$ where $e^{(i)} = 1$ if $y^{(i)} = 0, 1, \cdots (b-1)$, $e^{(i)} = 0$ otherwise.

We shall assume that a one to one correspondence has been established between the set of all positive integers λ and the set of all sets of integers $r_1, r_2, \cdots r_\mu, n$, such that $0 < r_1 < r_2 < \cdots < r_\mu \leqq n$. Then μ is defined as a function of λ. Let

$$U \equiv U(x, y, \lambda) = [(K \subset x) \subset x_{r_1,n}] \cdot [(K \subset x) \subset x_{r_2,n}] \cdot [(K \subset x) \subset x_{r_\mu,n}]$$

$$V \equiv V(x, y, \lambda) = \overline{\lim_{m\to\infty}} \mid p_m[U(x, y, \lambda)] - p_\lambda \mid$$

where $p_\lambda = (b/a)^\mu$ and $q_\lambda = 1 - p_\lambda$.

We shall prove that $V = 0$ almost everywhere in the region $\Delta: 0 < y < 1$. We have the equation

$$E(V \neq 0) = E(V > \tfrac{1}{2}) + E(V > \tfrac{1}{3}) + E(V > \tfrac{1}{4}) + \cdots.^{8}$$

Hence, it will be sufficient to prove that $m[E(V > \epsilon)] = 0$ for every positive number ϵ. Since

$$E(V > \epsilon) = \lim_{m_0 \to \infty} \sum_{m=m_0}^{\infty} E[\,|\, p_m(U) - p_\lambda \,|> \epsilon],$$

it follows that

$$m[E(V > \epsilon)] \leqq \sum_{m=m_0}^{\infty} m\{E[\,|\, p_m(U) - p_\lambda \,| > \epsilon]\} \text{ for every } m_0.$$

This will imply

$$m[E(V > \epsilon)] \leqq \lim_{m_0 \to \infty} \sum_{m=m_0}^{\infty} m\{E[\,|\, p_m(U) - p_\lambda \,| > \epsilon\}.$$

We shall prove the convergence of this series. In order to do this, we have to compute the measure of the set $E[\,|\, p_m(U) - p_\lambda \,| > \epsilon]$. The expression $p_m(U)$ depends upon m digits of U and hence upon $m\mu$ digits selected from the first mn digits of $K \subset x$. The mn digits of $K \subset x$ are in turn selected from digits of K by means of the selection x. If ν is an integer such that $\nu \cdot p_\nu(x) = mn$, then the first mn digits of $K \subset x$ are selected from the first ν digits of K. Thus $p_m(U)$ is determined by the first ν digits of K and hence by the first ν digits of y.

The measure of the set of points y for which the first ν digits are prescribed, is $a^{-\nu}$. Our problem resolves itself into the counting of the number of permutations of the first ν digits of y which give rise to a U such that $|\, p_m(U) - p_\lambda \,| > \epsilon$.

Of the first ν digits of y, only mn are used in determining the first mn digits of $K \subset x$. Each of the remaining $\nu - mn$ digits has a possible values. Hence, there are $a^{(\nu - mn)}$ ways of selecting those digits which are not utilized. Of the first mn digits of $K \subset x$, there are only $m\mu$ digits which are used. The remaining $m(n - \mu)$ digits of K correspond to $m(n - \mu)$ digits of y which can be selected in $a^{m(n-\mu)}$ ways. Given a specified set of s digits of U which are equal to 1, and the remaining $n - s$ digits equal to 0, if $|\, p_m(U) - p_\lambda \,| > \epsilon$, then $|\, s/m - p_\lambda \,| > \epsilon$. The s digits of this specified set correspond to μs digits of y, each of which can be given b possible values. Hence, these digits of y can be selected in $b^{\mu s}$ ways. A digit of U which is 0, corresponds to μ digits of y which can be selected in $a^\mu - b^\mu$ ways. Thus there are $(a^\mu - b^\mu)^{m-s}$ ways in which the $\mu(m - s)$ digits of y can be chosen so that the specified set of $m - s$ digits of U will be 0. The s digits of U

[8] $E(V \neq 0)$ is the set of points for which $V \neq 0$.

which are equal to 1, can be selected in ${}_mC_s$ ways and s can take on all values consistent with the relations $| s/m - p_\lambda | > \epsilon$ and $0 \leqq s \leqq m$. Therefore

$$m\{E[p_m(U) - p_\lambda \mid > \epsilon]\} = \sum_{|s/m-p_\lambda| > \epsilon} {}_mC_s\, b^{\mu s}(a^\mu - b^\mu)^{m-s} a^{m(n-\mu)} a^{\nu - mn} a^{-\nu}$$
$$= \sum_{|s/m-p_\lambda| > \epsilon} {}_mC_s\, p_\lambda{}^s\, q_\lambda{}^{m-s}.$$

Since the series $\sum_{m=1}^{\infty} \sum_{|s/m-p_\lambda| > \epsilon} {}_mC_s\, p_\lambda{}^s\, q_\lambda{}^{m-s}$ converges,[9] it follows that

$m[E(V > \epsilon)] = 0$, and hence $V = 0$ almost everywhere in Δ.

Let the set D consist of the selections $x_1, x_2, \cdots$ and let x_0 be the identity selection. Thus $x_0 = 1 = 1, 1, 1, 1, \cdots$. We shall let the set M consist of the numbers K which correspond to the numbers y belonging to the set

$$C\{ \sum_{i=0}^{\infty} \sum_{\lambda=1}^{\infty} E[V(x_i, y, \lambda) \neq 0]\} \qquad (C \text{ meaning "complement"})$$

Then every selection belonging to D leaves invariant the properties of every admissible number belonging to M. Since the measure of the set

$$\sum_{i=0}^{\infty} \sum_{\lambda=1}^{\infty} E[V(x_i, y, \lambda) \neq 0] \text{ is } 0,$$

the measure of its complement is 1. The set of numbers y associated with a given number K is of measure 0, and hence, the set M cannot be denumerable. If in particular, $a = 2$ and $b = 1$, then $K = 1 - y$ and the set M has the power of the continuum.

Theorem 1 shows that if the fundamental set be increased in such a way that it remains denumerable, this increase will not alter the set of admissible numbers appreciably. It also shows that in general the properties of an admissible number are not altered by the operation of selection. This fact will be brought out from another point of view by theorem 2.

The resultant of the operations of two selections on a collective is equivalent to the operation of a single selection on that collective. A fundamental set should be so chosen that the resultant of any two selections of the set, is itself a selection of the set. The fundamental set for admissible numbers possesses this group property.[10] It should be observed that if a fundamental set is increased, the set of collectives is not necessarily decreased. For example, the fundamental set for admissible numbers can be increased so as to include all rational selections.[11]

[9] For the proof of the convergence of this series, see Borel I, Chapitre I (Nombres normaux).

[10] A selection does not in general possess a unique inverse, and hence these transformations do not form a group.

[11] See Copeland I, theorem 17.

In theorem 2, x, y, K, $U(x,y,\lambda)$, and $V(x,y,\lambda)$ will be defined in the same manner as in theorem 1.

THEOREM 2. *For almost every y, $K \subset x$ is a member of the set $A(b/a)$ for almost every x (where $A(b/a)$ is the set of admissible numbers associated with the probability b/a).*

We shall prove first that $V(x,y,\lambda) = 0$ almost everywhere in the region Δ: $0 < x < 1$, $0 < y < 1$. We have to prove the convergence of the series

$$\sum_{m=m_0}^{\infty} m\{E[\,|\,p_m(U) - p_\lambda\,| > \epsilon]\}.$$

We have the relation

$$E[\,|\,p_m(U) - p_\lambda\,| > \epsilon] < E[p_{3mn}(x) \geqq 1/3] \cdot E[p_m(U) - p_\lambda\,| > \epsilon] + E[p_{3mn}(x) < 1/3].$$

If $p_{3mn}(x) \geqq 1/3$, then there exists an integer, ν, such that $\nu p_\nu(x) = mn$ and $\nu \leqq 3mn$. Since $m\{E[p_{3mn}(x) \geqq 1/3]\} \leqq 1$, we have the inequality

$$m\{E[\,|\,p_m(U) - p_\lambda\,| > \epsilon]\} \leqq \sum_{|s/m - p_\lambda| > \epsilon} {}_mC_s\, p_\lambda{}^s\, q_\lambda{}^{m-s} + \sum_{\nu < mn} {}_{3mn}C_\nu \cdot 2^{-3mn}.$$

Hence, the series converges and $V = 0$ almost everywhere in Δ.

Let $E = \sum_{\lambda=1}^{\infty} E[V(x,y,\lambda) \neq 0]$. Then $m(E) = 0$ and E can be included in a set E' of Borel measure 0. Let $\phi(x,y)$ be the characteristic function of the set E'.[12] Then [13]

$$0 = \iint_\Delta \phi(x,y)\,dx\,dy = \int_0^1 dx \int_0^1 \phi(x,y)\,dy.$$

Therefore, $\int_0^1 \phi(x,y)\,dx = 0$ for almost every y, and if y is such that $\int_0^1 \phi(x,y)\,dx = 0$, then $\phi(x,y) = 0$ for almost every x. Thus for almost every y, $K \subset x$ is a member of the set $A(b/a)$ for almost every x.

It is easily seen that almost every point of Δ is such that K is a member of the set $A(b/a)$, and hence,

THEOREM 3. *For almost every y, $K \subset x$ is a member of the set $A(b/a)$ and K is a member of the set $A(b/a)$ for almost every x.*

The following theorem is a corollary of theorem 3.

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[12] See de la Vallée Poussin I.

[13] See de la Vallée Poussin II.

THEOREM 4. *There exists a set of selections E and a set of admissible numbers M, such that E has the measure 1 and M is at least denumerable and the properties of every admissible number of M are invariant under every selection of E.*

Next, we shall show that both the set of selections E and the corresponding set of admissible numbers M can be non-denumerable. Let us assume that the numbers in the interval from 0 to 1 can be well ordered. Let us consider two such well ordered series and let one of them be called the selection series and the other, the admissible number series. The admissible number series shall contain all of the admissible numbers and no numbers which are not admissible. Consider the first member of the selection series. If the set of admissible numbers whose properties are not invariant under the operation of this selection, is not of measure 0, then this selection will be deleted from the series. Otherwise, we shall delete from the admissible number series all members whose properties are not invariant under the operation of the first member of the selection series. In either case, the remaining series will still be well ordered. Next let us consider the first element in the new admissible number series. This element will de deleted, if the set of selections which do not leave the properties of this element invariant is not of measure 0. Otherwise we shall delete from the selection series those selections which do not leave the properties invariant. This process will be continued, alternating between the selection series and the admissible number series. The order of procedure is determined, except for those elements which have no immediate predecessors. In the case of these elements, we shall perform the elimination for the selection series first. This process can not terminate in a denumerable number of steps. Hence we have

THEOREM 5. *There exists a set of selections E and a set of admissible numbers M, such that E and M are both nondenumerable, and the properties of every admissible number of M are invariant under the operation of every selection of E.*

It will be recalled that if K is admissible, then the numbers $K \subset x_{1,n}$, $K \subset x_{2,n}, \cdots K \subset x_{n,n}$ are independent. We shall consider to what extent this independence can be generalized. We shall, however, restrict ourselves to selections which are mutually exclusive.[14] This question can be investigated by means of the following device. Let x be expressed in the scale of n, i. e.,

[14] Two selections (considered as sequences) are mutually exclusive if their product is the sequence $0, 0, 0, \cdots$

$$x = x^{(1)}, x^{(2)}, x^{(3)}, \cdots \text{ where } x^{(k)} = 0, 1, 2, \cdots n-1 \qquad (k = 1, 2, \cdots).$$

Let

$$v_i = v_i^{(1)}, v_i^{(2)}, v_i^{(3)}, \cdots \text{ where } v_i^{(k)} = \begin{cases} 1 \text{ if } x^{(k)} = i-1, \\ 0 \text{ otherwise.} \end{cases}$$

Then the numbers $v_1, v_2, \cdots v_n$ represent n mutually exclusive selections. Given an admissible number K, we shall investigate the independence of the numbers $K \subset v_1, K \subset v_2, \cdots K \subset v_n$. We shall define the admissible number K by means of the equations

$$y = y^{(1)}, y^{(2)}, y^{(3)}, \cdots, \text{ where } y^{(i)} = 0, 1, 2, \cdots a-1,$$

and

$$K = e^{(1)}, e^{(2)}, e^{(3)}, \cdots, \text{ where } e^{(i)} = \begin{cases} 1 \text{ if } y^{(i)} = 0, 1, \cdots b-1, \\ 0 \text{ otherwise.} \end{cases}$$

Let

$$U(x, y, \lambda) = (K \subset v_{r_1}) \cdot (K \subset v_{r_2}) \cdots (K \subset v_{r_\mu})$$
$$V(x, y, \lambda) = \overline{\lim_{m \to \infty}} \mid p_m[U(x, y, \lambda) - p_\lambda \mid.$$

The scale n, in which x is expressed, depends upon λ, but x itself will be considered independent of λ. If $V(x, y, \lambda) = 0$, then the corresponding number K is said to satisfy a *generalized condition of admissibility*. If the set $\sum_{\lambda=1}^{\infty} E[V(x, y, \lambda) \neq 0]$ is of 0 measure, then K is said to satisfy almost every generalized condition of admissibility. We shall prove the following theorem.

THEOREM 6. *There exists a nondenumerable set of numbers such that each number K of the set satisfies almost every generalized condition of admissibility.*

We have the relation

$$E[\mid p_m(U) - p_\lambda \mid > \epsilon] < E[\mid p_m(U) - p_\lambda \mid > \epsilon] \cdot \prod_{i=1}^{\mu} E[p_{2mn}(v_{r_i}) \geqq 1/2n] + \sum_{i=1}^{\mu} E[p_{2mn}(v_{r_i}) < 1/2n].$$

If $p_{2mn}(v_{r_i}) \geqq 1/2n$, then there exists a ν_i such that $\nu_i \cdot p_{\nu_i}(v_{r_i}) = m$ and $\nu_i \leqq 2mn$. Let ν be the largest of the integers $\nu_1, \nu_2, \cdots \nu_\mu$.
Then

$$m\{E[\mid p_m(U) - p_\lambda \mid > \epsilon]\} \leqq \sum_{|s/m - p_\lambda| > \epsilon} {}_mC_s \, p_\lambda^s \, q_\lambda^{m-s} + \sum_{s/2mn < 1/2n} {}_{2mn}C_s \, (1/n)^s \, (1 - 1/n)^{2mn-s}.$$

The remainder of the proof is similar to that of theorem 3.

The theorems which we have proved indicate the possible latitude of choice for the fundamental set of selections. We shall now discuss briefly Tornier's contribution to the problem raised by Kamke.

Tornier replaces the linear sequences of von Mises by square matrices of the form $(e_i^{(j)})$ where the indices i and j take on separately all positive integral values. In place of the selection operation for the collective, Tornier permits the selection of rows from his matrices. He demands the independence of these rows where independence has the connotation previously mentioned in this paper. Tornier proves the consistency of such matrices. The disadvantage in his theory lies in the fact that it is not as accurate a picture of a set of physical measurements. In a set of measurements, linear order is indicated by time. A square array is formed only by mathematical artifice. Tornier's matrices are important in the following type of interpretation. Each row of such a matrix can be regarded as the sequence of measurements by a given experimenter. However, for such an interpretation, we should expect that each row of a matrix should possess properties similar to those of the collective. Such a matrix can be constructed. I have proved the existence of a set I of independent admissible numbers such that for every $p(0 < p < 1)$, $I \cdot A(p)$ has the power of the continuum.[15] The set of sequences I constitutes a matrix with a denumerable number of columns and a continuum of rows. Any ordered denumerable subset of I constitutes a Tornier matrix. The matrix I possesses the further property that its rows are collectives whose fundamental set consists of the selections $x_{r,n}$. The label space for I is, of course, the restricted space for admissible numbers. In a recent paper I have constructed a Tornier matrix such that the label space for each row has the power of the continuum, the probability distributions being given by Stieltjes integrals.[16]

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DEFINITION OF POST'S GENERALIZED NEGATIVE AND MAXIMUM IN TERMS OF ONE BINARY OPERATION.

By Donald L. Webb.

In 1921 Post[1] demonstrated that it was possible to construct a function for any order table in a system of m truth-values by the use of two primitive functions, $\sim_m p$ and $p \vee_m q$ which are generalizations of the functions $\sim p$ and $p \vee q$ in the two-valued case. Recently we[2] have been able to show that a function on m truth-values for any order table can be constructed in terms of one binary operation, using in this demonstration a negative that corresponds to Post's $\sim_m p$, a binary operator $p \, \alpha_m \, q$ which, for the value combinations used in the interpolation formula, corresponds to Post's $p \vee_m q$, and a binary operator $p \mid q$ which has no equivalent among the operators employed by Post. In the latter paper all operators were defined in terms of $p \mid q$. In this paper by redefining the truth-table of $p \mid q$ we are enabled to define Post's $\sim_m p$ and $p \vee_m q$ in terms of the " $\mid$ " function, thus greatly simplifying the proof that any m-valued logic can be generated by one binary operation. We find too that $p \mid q$ as so defined reduces in the two-valued case to one of Sheffer's functions,[3] as it evidently must.

The notation used in this paper is patterned after that of Post so as to avoid confusion.

Let $t_0, t_1, \cdots, t_{m-1}$, where m is any positive integer, signify the m truth values that an elementary proposition can assume in a m-valued logic. Denote by p, q elementary propositions. Let $p = t_i$ signify that the proposition p has the truth-value t_i. Make the two additional arithmetical definitions:

$$\begin{aligned} \min(i,j) &= i \quad \text{if} \quad i \leqq j \qquad\qquad (i,j = 0,1,2,\cdots) \\ &= j \quad \text{if} \quad i \geqq j; \\ i \equiv i_n \bmod n, &\quad (i = 0,1,2,\cdots) \qquad 0 \leqq i_n < n. \end{aligned}$$

Hence, $p \mid q$ is defined: if $p = t_i$, $q = t_j$ $(i, j = 0, 1, \cdots, m-1)$, then $p \mid q = t_k$ where $k = [\min(i,j) + 1]_m$.

[1] E. L. Post, *American Journal of Mathematics*, vol. 43 (1921), pp. 163-185.

[2] D. L. Webb, *Proceedings of the National Academy of Sciences*, vol. 21 (1935), pp. 252-254.

[3] H. M. Sheffer, *Transactions of the American Mathematical Society*, vol. 14 (1913), pp. 481-488.

THEOREM 1. $\sim_m p \equiv p \mid p$.

If $p = t_i$, then $p \mid p = t_k$ where $k = (i+1)_m$. Thus $p \mid p$ cyclically permutes the truth-values t_i, giving $p \mid p$ and $\sim_m p$ the same truth-table. Therefore the two are equivalent.

Using Post's definition, $\sim_m^2 p = \sim_m \sim_m p$, etc., we may write

THEOREM 2. $p \vee_m q \equiv \sim_m^{m-1} (p \mid q)$.

By repeating the above process we find that if $p = t_i$, $\sim_m^h p = t_k$, where $k = (i+h)_m$ $(h = 2, 3, \cdots, m-1)$. Hence, if $p = t_i$, $q = t_j$, then $\sim_m^{m-1} (p \mid q) = t_k$ where $k = \{[\min(i,j)+1]_m + m - 1\}_m$, or $k = \min(i,j)$. But $p \vee_m q$[4] as given by Post has the same truth-table, making the two equivalent.

Since Post has shown that we can generate a function of any order in a m-valued truth system by means of $\sim_m p$ and $p \vee_m q$, then, by using the above theorems, we can generate a function of any order in a m-valued truth system in terms of " $\mid$ ".

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[4] This is called a maximum since the higher truth-value has the smaller subscript.

AN OPERATIONAL SOLUTION OF THE MAXWELL FIELD EQUATIONS.[1]

By E. P. Northrop.

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1. *Introduction.* In empty space, containing no charges or currents, the classical electrodynamic field equations can, by a proper choice of units, be expressed as follows:

$$
\begin{aligned}
&\nabla \cdot \boldsymbol{e} = 0, \qquad &&\nabla \cdot \boldsymbol{h} = 0 \\
&\nabla \times \boldsymbol{e} = -\,\partial \boldsymbol{h}/\partial t, \qquad &&\nabla \times \boldsymbol{h} = \partial \boldsymbol{e}/\partial t.
\end{aligned} \tag{1.1}
$$

Here $\boldsymbol{e}$ is the electric intensity, and $\boldsymbol{h}$ the magnetic intensity. In the present paper we are concerned for the most part with the two equations involving the curls of $\boldsymbol{e}$ and $\boldsymbol{h}$. The significance of the other two equations will be discussed toward the end of the paper.

It is convenient for our purposes to think of the electromagnetic field as characterized by the six-component vector $\boldsymbol{v} = (e_x, e_y, e_z,\ h_x, h_y, h_z)$. This enables us to write the two curl equations of (1.1) as the single matrix equation

$$
\partial \boldsymbol{v}/\partial t = i\boldsymbol{H}\boldsymbol{v}, \tag{1.2}
$$

where $\boldsymbol{H}$ is the matrix operator

$$
\boldsymbol{H} \equiv \left\{
\begin{array}{c|ccc}
 & 0 & -\dfrac{1}{i}\dfrac{\partial}{\partial z} & \dfrac{1}{i}\dfrac{\partial}{\partial y} \\
0 & \dfrac{1}{i}\dfrac{\partial}{\partial z} & 0 & -\dfrac{1}{i}\dfrac{\partial}{\partial x} \\
 & -\dfrac{1}{i}\dfrac{\partial}{\partial y} & \dfrac{1}{i}\dfrac{\partial}{\partial x} & 0 \\
\hline
\begin{array}{c}\text{upper right,}\\ \text{signs changed}\end{array} & & 0 &
\end{array}
\right\} \tag{1.3}
$$

and $\boldsymbol{v}$ is written as a one-column matrix. The solution of (1.2) is formally

$$
\boldsymbol{v} = e^{it\boldsymbol{H}}\boldsymbol{v}_0, \tag{1.4}
$$

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[1] Presented to the Society September 6, 1934.

where v_0 is the vector giving the initial state of the field. That is, the vector characterizing the electromagnetic field at any point and at any time can be expressed as an operator applied to v_0. The purpose of this paper is to obtain rigorously an explicit integral form for the solution (1.4) of the equation (1.2). The treatment of the problem, and the terminology to be used will be based upon definitions and methods devised by M. H. Stone and others; and constant reference will be made to Stone's treatise "Linear Transformations in Hilbert Space," (*American Mathematical Society Colloquium Publications*, vol. 15, 1932). We shall refer throughout to this as simply "Stone."[2]

The reader, if unacquainted with the terminology of this work, would do well to refer to it for the definitions of the following terms: linear manifold (Definition 1.3), linear manifold determined by a set (1.4), transformation (2.1), extension of a transformation (2.2), adjoint of a transformation (2.8), symmetric transformation (2.9), self-adjoint transformation (2.11), essentially self-adjoint transformation (2.12), and unitary transformation (2.18). The following theorems are also of basic importance: Theorems 1.24, 1.25, 2.2, 2.6, 2.16, and 3.10. In addition, the discussion on unitary invariance at the end of the second chapter is worthy of attention.

It is perhaps advisable to make a few remarks in connection with these references. The space of functions which we shall use is the space $L_{2,6}$, composed of all vector point-functions f with components $(f_1, \cdots, f_6)$ defined over the whole of Euclidean space of three dimensions, and belonging to L_2.[3] The operations $+$ and $\cdot$ are defined as vector addition and scalar multiplication, the null element is defined to be $(0, \cdots, 0)$; and the function (f, g) is determined by the equation

$$(f, g) = \int\int\int_{-\infty}^{+\infty} (f_1 \bar{g}_1 + \cdots + f_6 \bar{g}_6)\, dx dy dz,$$

where the bar denotes complex conjugate. This space is a special case of the space which is shown to be a Hilbert space in Theorem 1.25. The norm of a

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[2] The author's attention has been called to a series of three articles by G. Herglotz in the *Berichte der Sächsischen Akademie*, vols. 78 (1926) and 80 (1928). In these articles (see in particular section 11, part III), methods are devised which, if modified and applied to the two curl equations of (1.1), appear to lead to results similar to those obtained in the present paper. These methods, however, neglect completely the questions of convergence and of domains of applicability of the various operators employed.

[3] It should be clearly understood that the functions to be considered in this article are functions of the time, t, as well as of the space coördinates x, y, z. The variable t, however, will be suppressed; and we shall write simply $f(x, y, z)$, etc.

function is denoted by $|f|$, and is defined as $(f, f)^{1/2}$. Throughout the paper we shall mean $\int_{-A}^{A}\int_{-A}^{A}\int_{-A}^{A}$ when we write $\int\int_{-A}^{A}\int$.

The Fourier transformation is introduced in Theorem 3. 10, but in a rather general way. The form which we shall have occasion to use can be described briefly as follows. Let T denote the Fourier transformation, and let $f(x, y, z) \varepsilon L_2$. Then

$$Tf(x, y, z) = \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{(2\pi)^{3/2}} \int\int_{-A}^{A}\int e^{i(x\xi+y\eta+z\zeta)} f(\xi, \eta, \zeta)\, d\xi d\eta d\zeta.$$

The symbol l.i.m. signifies as usual the limit in the mean (here of order 2). That is, $f(x, y, z) = \underset{n\to\infty}{\text{l.i.m.}} f_n(x, y, z)$ if $|f - f_n| \to 0$ as $n \to \infty$. Recall that if $f \varepsilon L_2$, then Tf and $T^{-1}f$ also belong to L_2. $T^{-1}f$ is defined by the same expression as above, save that the kernel is replaced by its complex conjugate.

2. *Outline of procedure.* Our problem, as stated in the last section, is to determine the operator $\boldsymbol{F}(\boldsymbol{H}) \equiv e^{it\boldsymbol{H}}$. A brief outline of the procedure to be followed may be of aid in understanding what will later be taken up in detail.

If $\boldsymbol{H}$ is a self-adjoint transformation, then $\boldsymbol{F}(\boldsymbol{H})$ is determined [4] by means of the equation

$$(\boldsymbol{F}(\boldsymbol{H})f, g) = \int_{-\infty}^{+\infty} F(\lambda)\, d(\boldsymbol{E}(\lambda)f, g).$$

The transformation $\boldsymbol{E}(\lambda)$ appearing in this relation is obtained by means of a certain contour integral [5] involving the inverse $\boldsymbol{H}_l^{-1}$ of the transformation $\boldsymbol{H}_l \equiv \boldsymbol{H} - l\boldsymbol{I}$, where $\boldsymbol{I}$ is the identity transformation, and l is an arbitrary not-real number. Thus the normal procedure would seem to be as follows: given the self-adjoint transformation $\boldsymbol{H}$, calculate $\boldsymbol{H}_l$, $\boldsymbol{H}_l^{-1}$, $\boldsymbol{E}(\lambda)$ and finally, $\boldsymbol{F}(\boldsymbol{H})$. Due, however, to the difficulties involved in the manipulation of the partial differential operator $\boldsymbol{H}$, we shall follow a less direct route.

The transform of the operator $\boldsymbol{H}$ by the Fourier transformation—call it $\boldsymbol{T}$—leads to a relatively simple algebraic operator, which we shall denote by $\boldsymbol{H}'$. That is $\boldsymbol{H}' \equiv \boldsymbol{THT}^{-1}$; and by Stone, Theorem 2. 55, $\boldsymbol{H}'$ is self-adjoint. We then calculate in succession $\boldsymbol{H}'_l$, $\boldsymbol{H}'_l{}^{-1}$, and $\boldsymbol{E}'(\lambda)$. But $\boldsymbol{H}' \equiv \boldsymbol{THT}^{-1}$ implies [6] $\boldsymbol{E}'(\lambda) \equiv \boldsymbol{TE}(\lambda)\boldsymbol{T}^{-1}$ which is equivalent to $\boldsymbol{E}(\lambda) \equiv \boldsymbol{T}^{-1}\boldsymbol{E}'(\lambda)\boldsymbol{T}$. Hence we are enabled to calculate $\boldsymbol{E}(\lambda)$, and so, $\boldsymbol{F}(\boldsymbol{H})$.

[4] Stone, Theorem 6. 1.

[5] Stone, Theorem 5. 10.

[6] Stone, Theorem 7. 1.

3. *The operators* $\boldsymbol{H}$ *and* $\boldsymbol{H}'$. The matrix operator $\boldsymbol{H}$ has already been defined in (1.3). The unitary transformation by which we obtain the transform of $\boldsymbol{H}$ is the diagonal matrix $\boldsymbol{T}$ whose elements are T, the Fourier transformation. The inverse $\boldsymbol{T}^{-1}$ of $\boldsymbol{T}$ is obtained by replacing T by T^{-1} in $\boldsymbol{T}$. It is then easy to show formally that the transform of $\boldsymbol{H}$ by $\boldsymbol{T}$ is the operator

$$(3.1) \qquad \boldsymbol{H}' \equiv \boldsymbol{T}\boldsymbol{H}\boldsymbol{T}^{-1} \equiv \left\{ \begin{array}{c|c} 0 & \begin{matrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{matrix} \\ \hline \begin{matrix}\text{upper right,} \\ \text{signs changed}\end{matrix} & 0 \end{array} \right\}.$$

It is evident that this operator lends itself much more readily to manipulation than does $\boldsymbol{H}$. It is to be noted, however, that we have only formally defined $\boldsymbol{H}$ and $\boldsymbol{H}'$ since we have said nothing of their domains.

It is relatively easy to find a domain in which $\boldsymbol{H}'$ is self-adjoint. On the other hand, the problem of showing that $\boldsymbol{H}$ is the inverse transform of $\boldsymbol{H}'$ (i. e., $\boldsymbol{H} \equiv \boldsymbol{T}^{-1}\boldsymbol{H}'\boldsymbol{T}$, in which case the self-adjointness of $\boldsymbol{H}$ is established, and its domain determined) presents complications. Let us indicate a possible method of solving this problem in the form of a theorem.

THEOREM I.

Hypothesis. 1.) *Let* $\boldsymbol{H}'$ *be a self-adjoint transformation with domain* $D(\boldsymbol{H}')$. 2.) *Denote* $\boldsymbol{H}'$ *restricted to a domain* $D(\boldsymbol{H}'_0) \subset D(\boldsymbol{H}')$ *by* $\boldsymbol{H}'_0$; *and let* $\boldsymbol{H}'_0$ *be essentially self-adjoint.* 3.) *Define* $\boldsymbol{H}_0$ *by means of the relations* $\boldsymbol{H}_0 \equiv \boldsymbol{T}^{-1}\boldsymbol{H}'_0\boldsymbol{T}$, $D(\boldsymbol{H}_0) \equiv \boldsymbol{T}^{-1}D(\boldsymbol{H}'_0)$. 4.) *Denote the adjoint of* $\boldsymbol{H}_0$ *by* $\boldsymbol{H}^*_0$ *and let* $\boldsymbol{H}^*_0 \equiv \boldsymbol{H}$, *where the domain of* $\boldsymbol{H}$ *is* $D(\boldsymbol{H})$.

Conclusion. $\boldsymbol{H}$ *is self-adjoint, and* $\boldsymbol{H} \equiv \boldsymbol{T}^{-1}\boldsymbol{H}'\boldsymbol{T}$ *throughout*

$$D(\boldsymbol{H}) \equiv \boldsymbol{T}^{-1}D(\boldsymbol{H}').$$

Proof. Since $\boldsymbol{H}'_0$ is essentially self-adjoint, so also is $\boldsymbol{H}_0$. Hence $\boldsymbol{H}^*_0 \equiv \boldsymbol{H}$ is self-adjoint. But by Stone, Theorem 2.53, $\boldsymbol{H}^*_0 \equiv \boldsymbol{T}^{-1}\boldsymbol{H}'^*_0\boldsymbol{T} \equiv \boldsymbol{T}^{-1}\boldsymbol{H}'\boldsymbol{T}$. Consequently $\boldsymbol{H} \equiv \boldsymbol{T}^{-1}\boldsymbol{H}'\boldsymbol{T}$ throughout its domain $D(\boldsymbol{H}) \equiv \boldsymbol{T}^{-1}D(\boldsymbol{H}')$ as we wished to show.

I have not been able to use this theorem to characterize $\boldsymbol{H}$ intrinsically, because of my inability to determine the adjoint $\boldsymbol{H}^*_0$ of $\boldsymbol{H}_0$ either directly or indirectly. Let us, however, leave this problem for later consideration, and turn our attention to the investigation of $\boldsymbol{H}'$, $\boldsymbol{H}'_0$, and $\boldsymbol{H}_0$.

4. *The operators* $\boldsymbol{H}'$ *and* $\boldsymbol{H}'_0$.

THEOREM II. *Let* $D(\boldsymbol{H}')$ *consist of all vector functions* $\boldsymbol{f} = (f_1, \cdots, f_6)$ *which belong to* $L_{2,6}$, *and with the property that the vector*

$$
(4.1) \qquad \boldsymbol{s} \equiv \left\{ \begin{array}{c|ccc} & 0 & z & -y \\ \mathbf{0} & -z & 0 & x \\ & y & -x & 0 \\ \hline \text{upper right,} & & & \\ \text{signs changed} & & \mathbf{0} & \\ & & & \end{array} \right\} \left\{ \begin{array}{c} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{array} \right\}
$$

also belongs to $L_{2,6}$. *Let* $\boldsymbol{H}'$ *be the transformation which takes* $\boldsymbol{f} \varepsilon D(\boldsymbol{H}')$ *into* $\boldsymbol{s}$. *Then* $\boldsymbol{H}'$ *is self-adjoint.*

Proof. It is evident that $D(\boldsymbol{H}')$ is a linear manifold. We prove first that $D(\boldsymbol{H}')$ is everywhere dense in $L_{2,6}$. Let D consist of all functions whose components can be expressed as linear combinations of functions which are defined as 1 inside and on an arbitrary axis-parallel parallelepiped, and zero elsewhere. Then $D \subset D(\boldsymbol{H}') \subset L_{2,6}$. But it is well known that D is dense in $L_{2,6}$. Hence also is $D(\boldsymbol{H}')$. That is, $D(\boldsymbol{H}')$ determines the closed linear manifold $L_{2,6}$. We can then prove that $\boldsymbol{H}'$ is symmetric by showing directly that the relation $(\boldsymbol{H}'\boldsymbol{f}, \boldsymbol{g}) - (\boldsymbol{f}, \boldsymbol{H}'\boldsymbol{g}) = 0$ is true for every $\boldsymbol{f}$ and $\boldsymbol{g}$ belonging to $D(\boldsymbol{H}')$. If the difference in question is written out, it will be found that the terms line up in pairs, cancelling each other, and that the desired result is obtained.

Now define $\boldsymbol{H}'_l$ by the relation $\boldsymbol{H}'_l = \boldsymbol{H}' - l\boldsymbol{I}$. Since $\boldsymbol{H}'$ is symmetric, $\boldsymbol{H}'_l$ has an inverse $\boldsymbol{H}'_l{}^{-1}$ whenever l is not real.[7] We shall prove that the domain of $\boldsymbol{H}'_l{}^{-1}$ is the entire space $L_{2,6}$. Consider the solution of the equation $\boldsymbol{H}'_l{}^{-1}\boldsymbol{f} = \boldsymbol{g}$ where $\boldsymbol{g}$ is an arbitrary element of $L_{2,6}$. In matrix form, this equation is

$$
\left\{ \begin{array}{ccc|ccc} -l & 0 & 0 & 0 & z & -y \\ 0 & -l & 0 & -z & 0 & x \\ 0 & 0 & -l & y & -x & 0 \\ \hline & \text{upper right,} & & & \text{upper} & \\ & \text{signs changed} & & & \text{left} & \\ & & & & & \end{array} \right\} \left\{ \begin{array}{c} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{array} \right\} = \left\{ \begin{array}{c} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \end{array} \right\}.
$$

[7] Stone, Theorem 4.14.

To solve this equation for f, we compute by ordinary means the inverse of H'_l. The solution is then $f = H'^{-1}_l g$, or, in matrix form,

$$(4.2)\quad \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{Bmatrix} = \frac{1}{l(l^2 - x^2 - y^2 - z^2)} \left\{ \begin{array}{ccc|ccc} x^2 - l^2 & xy & xz & 0 & -lz & ly \\ yx & y^2 - l^2 & yz & lz & 0 & -lx \\ zx & zy & z^2 - l^2 & -ly & lx & 0 \\ \hline & \text{upper right, signs changed} & & & \text{upper left} & \end{array} \right\} \begin{Bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \end{Bmatrix}.$$

It is apparent that, since l is not real, the effect of applying H'^{-1}_l to $g \in L_{2,6}$ is to multiply each component of g by a bounded, measurable function. It follows that $f \in L_{2,6}$. Now apply H'_l to both sides of (4.2). We have $H'f - lf = g$, or $H'f = lf + g$. Since the right-hand side of the last equation belongs to $L_{2,6}$, so also does the left-hand side, which is immediately identified as the vector s of (4.1). It follows that $f \in D(H')$. That is, H'^{-1}_l carries $L_{2,6}$ in a one-to-one manner into $D(H')$, this being true for all not-real l. This is equivalent to saying that the ranges of H'_{+i} and H'_{-i} are both $L_{2,6}$. H' is consequently self-adjoint.[8]

THEOREM III. *Let $D(H'_0)$ consist of all vector functions $f = (f_1, \cdots, f_6)$ which belong to $L_{2,6}$, and with the property that any of their components multiplied by x, y, or z belongs to L_2. Let H'_0 be the transformation which takes $f \in D(H'_0)$ into the vector s of (4.1). Then H'_0 is essentially self-adjoint.*

Proof. It can be shown (i) that $D(H'_0)$ determines the closed linear manifold $L_{2,6}$, and (ii) that H'_0 is symmetric by precisely the same method as that used in Theorem II.

We now determine directly the adjoint H'^*_0 of H'_0. Its domain consists of those and only those elements $g \in L_{2,6}$ such that the relation $(H'_0 f, g) = (f, g^*)$ holds for all $f \in D(H'_0)$ and some element $g^* \in L_{2,6}$; and, for such an element, $H'^*_0 g = g^*$. Consider the equation

$$(H'_0 f, g) = (f, g^*)$$

for functions f vanishing outside an arbitrary cube. We have

$$\int\int_{-A}^{A}\int [(zf_5 - yf_6)\bar{g}_1 + (-zf_4 + xf_6)\bar{g}_2 + (yf_4 - xf_5)\bar{g}_3$$
$$+ (-zf_2 + yf_3)\bar{g}_4 + (zf_1 - xf_3)\bar{g}_5 + (-yf_1 + xf_2)\bar{g}_6]dxdydz$$
$$= \int\int_{-A}^{A}\int [f_1\bar{g}^*_1 + f_2\bar{g}^*_2 + f_3\bar{g}^*_3 + f_4\bar{g}^*_4 + f_5\bar{g}^*_5 + f_6\bar{g}^*_6]dxdydz.$$

[8] Stone, Theorems 9.1 to 9.3.

Because of the arbitrariness of $\boldsymbol{f}$, we must have, almost everywhere,

$$\begin{aligned} g^*_1 &= \quad zg_5 - yg_6 & \qquad g^*_4 &= -zg_2 + yg_3 \\ g^*_2 &= -zg_4 + xg_6 & \qquad g^*_5 &= \quad zg_1 - xg_3 \\ g^*_3 &= \quad yg_4 - xg_5 & \qquad g^*_6 &= -yg_1 + xg_2. \end{aligned}$$

Since these relations hold in an arbitrary cube, they must hold over all space. $\boldsymbol{H'^*}_0$ is therefore a transformation with a domain consisting of all functions $\boldsymbol{g} \,\varepsilon\, L_{2,6}$ such that $\boldsymbol{g}^*$ also belongs to $L_{2,6}$, and which takes a function in its domain into $\boldsymbol{g}^*$ as above. $\boldsymbol{H'^*}_0$ is thus identified as the transformation $\boldsymbol{H'}$, which is self-adjoint. Hence $\boldsymbol{H'}_0$ is essentially self-adjoint.

5. *The operator* $\boldsymbol{H}_0 \equiv \boldsymbol{T}^{-1}\boldsymbol{H'}_0\boldsymbol{T}$.

THEOREM IV. *Let* $\boldsymbol{H}_0$ *be the transformation defined by means of the relation* $\boldsymbol{H}_0 \equiv \boldsymbol{T}^{-1}\boldsymbol{H'}_0\boldsymbol{T}$, *where* $\boldsymbol{H'}_0$ *is the transformation of Theorem III. Then the domain* $D(\boldsymbol{H}_0)$ *of* $\boldsymbol{H}_0$ *consists of all vector functions* $\boldsymbol{g} \,\varepsilon\, L_{2,6}$ *such that the components are absolutely continuous in* x, y, *and* z *separately and have the property that their first partial derivatives with respect to* x, y, *or* z *belong to* L_2. $\boldsymbol{H}_0$ *is the essentially self-adjoint transformation which takes* $\boldsymbol{g} \,\varepsilon\, D(\boldsymbol{H}_0)$ *into the vector defined by the expression*

$$\left\{ \begin{array}{c|c} 0 & \begin{matrix} 0 & -\frac{1}{i}\frac{\partial}{\partial z} & \frac{1}{i}\frac{\partial}{\partial y} \\ \frac{1}{i}\frac{\partial}{\partial z} & 0 & -\frac{1}{i}\frac{\partial}{\partial x} \\ -\frac{1}{i}\frac{\partial}{\partial y} & \frac{1}{i}\frac{\partial}{\partial x} & 0 \end{matrix} \\ \hline \begin{matrix}\text{upper right,} \\ \text{signs changed}\end{matrix} & 0 \end{array} \right\} \left\{ \begin{matrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \end{matrix} \right\}$$

Proof. Since $\boldsymbol{H'}_0$ has been shown to be essentially self-adjoint, $\boldsymbol{H}_0$ will also enjoy that property, by virtue of Stone, Theorem 2. 55. The domain of $\boldsymbol{H}_0$ is characterized as follows: if $\boldsymbol{H'}_0$ takes $\boldsymbol{f} \,\varepsilon\, D(\boldsymbol{H'}_0)$ into $\boldsymbol{f}^*$, then $\boldsymbol{H}_0$ takes $\boldsymbol{T}^{-1}\boldsymbol{f} \,\varepsilon\, D(\boldsymbol{H}_0)$ into $\boldsymbol{T}^{-1}\boldsymbol{f}^*$. If we put $\boldsymbol{g} \equiv \boldsymbol{T}^{-1}\boldsymbol{f}$, $\boldsymbol{g}^* \equiv \boldsymbol{T}^{-1}\boldsymbol{f}^* \equiv \boldsymbol{T}^{-1}[\boldsymbol{H'}_0\boldsymbol{f}]$, we must have

$$\begin{aligned} g^*_1 &= T^{-1}[\quad zf_6 - yf_5] & \qquad g^*_4 &= T^{-1}[-zf_2 + yf_3] \\ g^*_2 &= T^{-1}[-zf_4 + xf_6] & \qquad g^*_5 &= T^{-1}[\quad zf_1 - xf_3] \\ g^*_3 &= T^{-1}[\quad yf_4 - xf_5] & \qquad g^*_6 &= T^{-1}[-yf_1 + xf_2]. \end{aligned}$$

Because of the linearity of T^{-1}, and of the symmetry which manifests itself throughout, it will be sufficient if we put $g = T^{-1}f$, $g^* = T^{-1}[xf]$, and determine what g^* is in terms of g. We shall show first that

$$(5.1) \qquad g^*(x, y, z) = -\frac{1}{i} \underset{h\to 0}{\text{l.i.m.}} \frac{g(x+h, y, z) - g(x, y, z)}{h}.$$

We have

$$g(x, y, z) = \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{(2\pi)^{3/2}} \int\int\int_{-A}^{A} e^{-i(x\xi+y\eta+z\zeta)} f(\xi, \eta, \zeta)\, d\xi d\eta d\zeta,$$

$$g^*(x, y, z) = \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{(2\pi)^{3/2}} \int\int\int_{-A}^{A} e^{-i(x\xi+y\eta+z\zeta)} \xi f(\xi, \eta, \zeta)\, d\xi d\eta d\zeta.$$

It is well known that $|f| = |Tf| = |T^{-1}f|$. Hence

$$(5.2) \quad \left| \frac{g(x+h, y, z) - g(x, y, z)}{ih} + g^*(x, y, z) \right| = \left| \left(\frac{e^{-ih\xi} - 1}{ih} + \xi \right) f(\xi, \eta, \zeta) \right|.$$

Now it can be shown by virtue of the mean value theorem that

$$\left| \frac{e^{-ih\xi} - 1}{ih} \right| < (2)^{1/2} |\xi|,$$

and consequently that

$$\left| \left(\frac{e^{-ih\xi} - 1}{ih} + \xi \right) f(\xi, \eta, \zeta) \right| < k\, |\xi f(\xi, \eta, \zeta)|,$$

where k is a suitable constant. In addition,

$$\lim_{h\to 0} \left(\frac{e^{-ih\xi} - 1}{ih} + \xi \right) = 0.$$

These last two relations are sufficient, by a familiar theorem regarding passage to the limit under the sign of integration, to insure the convergence of the right-hand side of (5.2) to zero as $h \to 0$. This in turn implies (5.1), as we wished to show.

We propose now to show that

$$g^*(x, y, z) = -\frac{1}{i} \frac{\partial}{\partial x} g(x, y, z)$$

almost everywhere. Since g and g^* belong to L_2, we have by (5.1)

$$\lim_{h\to 0}\int_a^x\int_b^y\int_c^z \frac{g(\xi+h,\eta,\zeta)-g(\xi,\eta,\zeta)}{ih}\,d\xi d\eta d\zeta$$
$$=-\int_a^x\int_b^y\int_c^z g^*(\xi,\eta,\zeta)\,d\xi d\eta d\zeta.$$

That is,

$$\lim_{h\to 0}\frac{1}{h}\left[\int_x^{x+h}-\int_a^{a+h}\right]\int_b^y\int_c^z g(\xi,\eta,\zeta)\,d\xi d\eta d\zeta$$
$$=-i\int_a^x\int_b^y\int_c^z g^*(\xi,\eta,\zeta)\,d\xi d\eta d\zeta,$$

and

$$(5.3)\quad \int_b^y\int_c^z [g(x,\eta,\zeta)-g(a,\eta,\zeta)]\,d\eta d\zeta$$
$$=-i\int_a^x\int_b^y\int_c^z g^*(\xi,\eta,\zeta)\,d\xi d\eta d\zeta$$

for all x outside a one-dimensional null set dependent on y and z. To show that this relation holds for all x, y, z outside a three-dimensional null set, we argue as follows. Since the relation is one between measurable functions, the set on which it fails to hold is certainly a measurable set. Denote this three-dimensional set by S. The two-dimensional intersection S_z of S with $z=$ constant is measurable for all z outside a one-dimensional null set. For all such z, the one-dimensional intersection S_{zy} of S_z with $y=$ constant is measurable for all y outside a one-dimensional null set. But for all such y, the projection of S_{zy} on the axis of x is of measure zero, since (5.3) holds for all x outside a one-dimensional null set dependent on y and z. By a theorem of Fubini,[9] it follows that the two-dimensional set S_z is of measure zero. We need only repeat the argument to show that S is likewise of measure zero.

Now change the order of integration on the right-hand side of (5.3), and differentiate with respect to y. Then

$$\int_c^z [g(x,y,\zeta)-g(a,y,\zeta)]\,d\zeta=-i\int_c^z\int_a^x g^*(\xi,y,\zeta)\,d\xi d\zeta$$

for all y outside a one-dimensional null set dependent on x and z. We have only to repeat the argument used above to show that this relation holds for all x, y, z outside a three-dimensional null set. Now differentiate with respect to z.

$$g(x,y,z)-g(a,y,z)=-i\int_a^x g^*(\xi,y,z)\,d\xi$$

[9] See, e. g., C. Caratheodory, *Vorlesungen über Reele Funktionen* (1918), Satz 3, p. 628.

for all z outside a one-dimensional null set dependent on x and y. We can likewise show that this relation holds except in a three-dimensional null set. It is now evident that for a fixed y and z outside a certain two-dimensional null set, $g(x, y, z)$ is equal for almost all x to an absolutely continuous function of x, whose derivative is $-ig^*(x, y, z)$ for almost all x. A third and final repetition of the argument used above results in the conclusion that

$$g^*(x, y, z) = -\frac{1}{i}\frac{\partial}{\partial x} g(x, y, z)$$

almost everywhere.

It remains to show conversely that, if we put as before $f = Tg$, $f^* = Tg^*$, then $f^* = xf$. Let

$$f(x, y, z; A) = \frac{1}{(2\pi)^{3/2}} \int\int\int_{-A}^{A} e^{i(x\xi+y\eta+z\zeta)} g(\xi, \eta, \zeta)\, d\xi d\eta d\zeta,$$

$$f^*(x, y, z; A) = \frac{-1}{i(2\pi)^{3/2}} \int\int\int_{-A}^{A} e^{i(x\xi+y\eta+z\zeta)} \frac{\partial}{\partial \xi} g(\xi, \eta, \zeta)\, d\xi d\eta d\zeta.$$

Then $f(x, y, z; A)$ and $f^*(x, y, z; A)$ converge in the mean as $A \to \infty$ to $f(x, y, z)$ and $f^*(x, y, z)$ respectively. We can, as well as not, assume that A runs over the positive integers. It follows that there exists a subsequence of integers, say $\{m\}$, for which, almost everywhere, $f(x, y, z; m)$ and $f^*(x, y, z; m)$ converge in the ordinary sense to $f(x, y, z)$ and $f^*(x, y, z)$ respectively. If now we define the function $h(x, y, z)$ by the relation

$$h(x, y, z) \equiv f^*(x, y, z) - xf(x, y, z),$$

then the function

$$h(x, y, z; m) \equiv f^*(x, y, z; m) - xf(x, y, z; m)$$

tends almost everywhere to $h(x, y, z)$ as $m \to \infty$. In addition, $h(x, y, z)$ is integrable over every finite interval. We propose to show that $h(x, y, z) \equiv 0$ almost everywhere. If we integrate $f^*(x, y, z; m)$ by parts with respect to ξ, we have

$$h(x, y, z; m) = \frac{-1}{i(2\pi)^{3/2}} \int\int_{-m}^{m} \left[e^{i(x\xi+y\eta+z\zeta)} g(\xi, \eta, \zeta)\right]_{\xi=-m}^{\xi=m} d\eta d\zeta$$

$$= -\frac{e^{imx}}{i(2\pi)^{3/2}} \int\int_{-m}^{m} e^{i(y\eta+z\zeta)} g(m, \eta, \zeta)\, d\eta d\zeta$$

$$+ \frac{e^{-imx}}{i(2\pi)^{3/2}} \int\int_{-m}^{m} e^{i(y\eta+z\zeta)} g(-m, n, \zeta)\, d\eta d\zeta.$$

That is, we can write $h(x, y, z; m)$ in the form

$$h(x, y, z; m) = \phi(y, z; m) e^{imx} + \psi(y, z; m) e^{-imx}.$$

Now separate all quantities into their real and imaginary parts; i.e., put $h = h_1 + ih_2$, $\phi = \phi_1 + i\phi_2$, etc. The above relation then becomes

$$\begin{aligned} h_1(x, y, z; m) &= (\phi_1 + \psi_1) \cos mx + (-\phi_2 + \psi_2) \sin mx, \\ h_2(x, y, z; m) &= (\phi_2 + \psi_2) \cos mx + (\phi_1 - \psi_1) \sin mx, \end{aligned} \tag{5.4}$$

and in addition,

$$\lim_{m \to \infty} h_j(x, y, z; m) = h_j(x, y, z), \qquad (j = 1, 2), \tag{5.5}$$

almost everywhere.

If now we fix y and z, ϕ and ψ become functions of m alone, and the right-hand side of either of the relations of (5.4) can be written in the form $B_m \cos mx + C_m \sin mx$, where the coefficients B_m and C_m are real numbers dependent on m. By a theorem of Steinhaus,[10]

$$\overline{\lim_{m \to \infty}} \mid B_m \cos mx + C_m \sin mx \mid = \overline{\lim_{m \to \infty}} (B_m{}^2 + C_m{}^2)^{1/2}$$

for all x outside a null set. The left-hand side is finite for all x outside a null set (since h_1 and h_2 are integrable over every finite interval), and the right-hand side is independent of x. Hence B_m and C_m remain bounded. Now write

$$B_m \cos mx + C_m \sin mx = (B_m{}^2 + C_m{}^2)^{1/2} \cos m(x - \alpha_m),$$

where $\alpha_m = \tan^{-1} C_m/B_m$; and note that for almost all x,

$$\overline{\lim_{m \to \infty}} \cos m(x - \alpha_m) = +1, \qquad \underline{\lim_{m \to \infty}} \cos m(x - \alpha_m) = -1.$$

Furthermore, we can pick out a subsequence, say $\{\mu\}$ of m's such that $(B_\mu{}^2 + C_\mu{}^2)^{1/2}$ tends to its upper limit. In that case,

$$\overline{\lim_{\mu \to \infty}} (B_\mu{}^2 + C_\mu{}^2)^{1/2} \cos \mu(x - \alpha_\mu) = + \overline{\lim_{\mu \to \infty}} (B_\mu{}^2 + C_\mu{}^2)^{1/2} \geq 0,$$

$$\underline{\lim_{\mu \to \infty}} (B_\mu{}^2 + C_\mu{}^2)^{1/2} \cos \mu(x - \alpha_\mu) = - \overline{\lim_{\mu \to \infty}} (B_\mu{}^2 + C_\mu{}^2)^{1/2} \leq 0.$$

But the limits on the left are identical by (5.5). Consequently $h(x, y, z) \equiv 0$ almost everywhere, as we wished to show. This completes the proof of Theorem IV.

[10] *Wiadomosci Matematyczne*, vol. 24 (1920), pp. 197-201.

Let us consider what has been accomplished up to this point. We have defined the operators $\boldsymbol{H}'$, $\boldsymbol{H}'_0$, and $\boldsymbol{H}_0$; and have shown $\boldsymbol{H}'$ self-adjoint, and $\boldsymbol{H}'_0$ and $\boldsymbol{H}_0$ essentially self-adjoint. Were we able to determine the adjoint $\boldsymbol{H}^*_0$ of $\boldsymbol{H}_0$ we would have, as proved in Theorem I (Section 3), a self-adjoint transformation identical with $\boldsymbol{T}^{-1}\boldsymbol{H}'\boldsymbol{T}$ throughout its domain $\boldsymbol{T}^{-1}D(\boldsymbol{H}')$. As I remarked previously, I have been unable to determine $\boldsymbol{H}^*_0$ either directly or indirectly. For all practical purposes, however, the exact determination of $\boldsymbol{H}^*_0$ is unnecessary, since the functions to which one might have occasion to apply the theory would probably satisfy much more restrictive conditions than those required of functions in the domain of $\boldsymbol{H}^*_0$. Indeed, most of the functions considered in classical electrodynamics would be included in the domain of $\boldsymbol{H}_0$ (i. e., possess the necessary derivatives, belong to $L_{2,6}$, etc.). Consequently, since we are sure $\boldsymbol{H}^*_0$ exists, we shall hereafter refer to it as the self-adjoint transformation $\boldsymbol{H}$ with domain $D(\boldsymbol{H})$.

6. *The operator* $\boldsymbol{E}'(\lambda)$ *corresponding to* $\boldsymbol{H}'$. In the calculation of the operator $\boldsymbol{E}'(\lambda)$ we make use of Stone, Theorem 5. 10, to wit: If $\boldsymbol{H}'$ is a given self-adjoint transformation, the corresponding "resolution of the identity" $\boldsymbol{E}'(\lambda)$ can be determined from the relation

$$\begin{aligned}(6.1)\qquad &\tfrac{1}{2}\{[(\boldsymbol{E}'(\mu)\boldsymbol{f},\boldsymbol{g}) + (\boldsymbol{E}'(\mu-0)\boldsymbol{f},\boldsymbol{g})] \\ &\qquad - [(\boldsymbol{E}'(\nu)\boldsymbol{f},\boldsymbol{g}) + (\boldsymbol{E}'(\nu-0)\boldsymbol{f},\boldsymbol{g})]\} \\ &\qquad\qquad = -\frac{1}{2\pi i}\lim_{\epsilon\to 0}\int_C (\boldsymbol{H}'^{-1}_l\boldsymbol{f},\boldsymbol{g})\,dl,\end{aligned}$$

where $\boldsymbol{f}$ and $\boldsymbol{g}$ are arbitrary elements of $L_{2,6}$, and where the contour over which the integral is taken consists of two oriented polygonal lines whose vertices, in order, are $\mu + i\epsilon$, $\mu + i\alpha$, $\nu + i\alpha$, $\nu + i\epsilon$, and $\nu - i\epsilon$, $\nu - i\alpha$, $\mu - i\alpha$, $\mu - i\epsilon$, respectively; the real numbers μ, ν, α, ϵ, being subject to the inequalities $\nu < \mu$, $0 < \epsilon < \alpha$.

To obtain $\boldsymbol{E}'(\lambda)$ from (6. 1) we put $\mu = \lambda + \delta$, $\delta > 0$, and allow δ to tend to zero, and ν to tend to $-\infty$. For by the properties [11] of $\boldsymbol{E}'(\lambda)$,

$$\begin{gathered}\lim_{\delta\to 0}(\boldsymbol{E}'(\lambda+\delta)\boldsymbol{f},\boldsymbol{g}) = (\boldsymbol{E}'(\lambda+0)\boldsymbol{f},\boldsymbol{g}) = (\boldsymbol{E}'(\lambda)\boldsymbol{f},\boldsymbol{g}),\\ \lim_{\delta\to 0}(\boldsymbol{E}'(\lambda+\delta-0)\boldsymbol{f},\boldsymbol{g}) = (\boldsymbol{E}'(\lambda+0)\boldsymbol{f},\boldsymbol{g}) = (\boldsymbol{E}'(\lambda)\boldsymbol{f},\boldsymbol{g}),\\ \lim_{\nu\to-\infty}(\boldsymbol{E}'(\nu)\boldsymbol{f},\boldsymbol{g}) = 0, \qquad \lim_{\nu\to-\infty}(\boldsymbol{E}'(\nu-0)\boldsymbol{f},\boldsymbol{g}) = 0,\end{gathered}$$

and the left-hand side of (6. 1) becomes simply $(\boldsymbol{E}'(\lambda)\boldsymbol{f},\boldsymbol{g})$. It is to be noted that the domain of $\boldsymbol{E}'(\lambda)$ is the entire space $L_{2,6}$.

[11] Stone, Definition 5. 1.

For convenience of notation, let us write $(\boldsymbol{E}'(\mu,\nu)\boldsymbol{f},\boldsymbol{g})$ for the left-hand side of (6. 1), and put $\boldsymbol{R}\equiv\boldsymbol{H}'_l{}^{-1}$ (see relation (4. 2) for the latter). Denote their elements by $E'_{jk}(\mu,\nu)$ and R_{jk} respectively. Then (6. 1) becomes

$$(6.2)\qquad \iiint_{-\infty}^{+\infty} \sum_{j,k=1}^{6} E'_{jk}(\mu,\nu) f_k \bar{g}_j\, dxdydz$$
$$= -\frac{1}{2\pi i}\lim_{\epsilon\to 0}\int_C \left[\iiint_{-\infty}^{+\infty}\sum_{j,k=1}^{6} R_{jk} f_k \bar{g}_j\, dxdydz\right] dl.$$

The change of order of integration on the right-hand side of this equation can easily be justified because of the simple way in which l enters into the expression for the R_{jk}. The right-hand member can thus be written

$$(6.3)\qquad \lim_{\epsilon\to 0}\iiint_{-\infty}^{+\infty}\sum_{j,k=1}^{6}\left[-\frac{1}{2\pi i}\int_C R_{jk}dl\right] f_k\bar{g}_j\, dxdydz.$$

We proceed now to calculate the various contour integrals involved. Inspection of the matrix $\boldsymbol{R}$ shows that the calculation requires integrating the following three fractions, or combinations thereof, where for simplicity we put $r\equiv(x^2+y^2+z^2)^{1/2}$: $1/(l^2-r^2)$, $l/(l^2-r^2)$, $1/l(l^2-r^2)$. These in turn break down into partial fractions whose contour integrals are easily found. The following results are obtained:

$$-\frac{1}{2\pi i}\int_C\frac{dl}{l^2-r^2} = \frac{1}{2\pi r}\left[\tan^{-1}\frac{\nu-r}{\epsilon} - \tan^{-1}\frac{\mu-r}{\epsilon} - \tan^{-1}\frac{\nu+r}{\epsilon} + \tan^{-1}\frac{\mu+r}{\epsilon}\right]$$
$$-\frac{1}{2\pi i}\int_C\frac{ldl}{l^2-r^2} = \frac{1}{2\pi}\left[\tan^{-1}\frac{\nu-r}{\epsilon} - \tan^{-1}\frac{\mu-r}{\epsilon} + \tan^{-1}\frac{\nu+r}{\epsilon} - \tan^{-1}\frac{\mu+r}{\epsilon}\right]$$
$$-\frac{1}{2\pi i}\int_C\frac{dl}{l(l^2-r^2)} = \frac{1}{2\pi r^2}\left[\tan^{-1}\frac{\nu-r}{\epsilon} - \tan^{-1}\frac{\mu-r}{\epsilon} + \tan^{-1}\frac{\nu+r}{\epsilon} - \tan^{-1}\frac{\mu+r}{\epsilon}\right.$$
$$\left. - 2\tan^{-1}\frac{\nu}{\epsilon} + 2\tan^{-1}\frac{\mu}{\epsilon}\right].$$

Note now that as $\epsilon\to 0$ through positive values, these expressions converge boundedly to their respective limits. We are consequently justified in passing to the limit under the sign of integration in (6. 3). That is, if we put

$$P_{jk}\equiv -\frac{1}{2\pi i}\lim_{\epsilon\to 0}\int_C R_{jk}dl,$$

we can conclude that

$$\iiint_{-\infty}^{+\infty}\sum_{j,k=1}^{6} E'_{jk}(\mu,\nu) f_k\bar{g}_j\, dxdydz = \iiint_{-\infty}^{+\infty}\sum_{j,k=1}^{6} P_{jk} f_k\bar{g}_j\, dxdydz.$$

Finally, in view of the fact that f and g are arbitrary elements of $L_{2,6}$, we must have $E'_{jk}(\mu,\nu)f_k = P_{jk}f_k$ almost everywhere.

In calculating the limits of the contour integrals, there are three cases to be considered, namely, (i) $0 < \nu < \mu$, (ii) $\nu < 0 < \mu$, (iii) $\nu < \mu < 0$. (Recall that μ was taken greater than ν). But since, in passing from $E'_{jk}(\mu,\nu)$ to $E'_{jk}(\lambda)$, we are going to let ν tend to $-\infty$, we may as well take ν negative, which throws out case (i). Also, in case (ii), we shall assume $\mu < -\nu$. If, then, we put $\mu = \lambda + \delta$, $\delta > 0$, and allow δ to tend to zero, ν to tend to $-\infty$, $E'_{jk}(\mu,\nu)$ tends in the mean to $E'_{jk}(\lambda)$, and we obtain the following matrices:

$$\underset{(\lambda<0)}{E'(\lambda)} = \begin{cases} \boldsymbol{O} \quad \text{(null matrix)}, & r \leq -\lambda \\[1ex] \dfrac{-1}{2r^2}\left\{\begin{array}{ccc|ccc} x^2-r^2 & xy & xz & 0 & zr & -yr \\ yx & y^2-r^2 & yz & -zr & 0 & xr \\ zx & zy & z^2-r^2 & yr & -xr & 0 \\ \hline & \text{upper right, signs changed} & & & \text{upper left} & \end{array}\right\}, & r > -\lambda \end{cases}$$

$$\underset{(\lambda>0)}{E'(\lambda)} = \begin{cases} \boldsymbol{I} \quad \text{(identity matrix)}, & r \leq \lambda \\[1ex] \dfrac{1}{2r^2}\left\{\begin{array}{ccc|ccc} x^2+r^2 & xy & xz & 0 & -zr & yr \\ yx & y^2+r^2 & yz & zr & 0 & -xr \\ zx & zy & z^2+r^2 & -yr & xr & 0 \\ \hline & \text{upper right, signs changed} & & & \text{upper left} & \end{array}\right\}, & r > \lambda. \end{cases}$$

The operators which for the most part we shall have occasion to use, however, are not those just calculated, but rather the operators $[E'(\lambda) - E'(\mu)]$, as they appear in the cases $\mu < \lambda < 0$ and $0 < \mu < \lambda$. If we choose the following nine elements as basic:

$$E'_{11} = \frac{r^2-x^2}{2r^2} \qquad E'_{12} = \frac{-xy}{2r^2} \qquad E'_{35} = \frac{-x}{2r}$$

$$E'_{22} = \frac{r^2-y^2}{2r^2} \qquad E'_{13} = \frac{-xz}{2r^2} \qquad E'_{16} = \frac{-y}{2r}$$

$$E'_{33} = \frac{r^2-z^2}{2r^2} \qquad E'_{23} = \frac{-yz}{2r^2} \qquad E'_{24} = \frac{-z}{2r},$$

we can express the required $[E'(\lambda) - E'(\mu)]$ in terms of them as follows:

$$
\begin{array}{c}
[\boldsymbol{E}'(\lambda)-\boldsymbol{E}'(\mu)]= \\
(\mu<\lambda<0)
\end{array}
\left\{
\begin{array}{l}
\left\{
\begin{array}{ccc|ccc}
E'_{11} & E'_{12} & E'_{13} & 0 & E'_{24} & -E'_{16} \\
E'_{12} & E'_{22} & E'_{23} & -E'_{24} & 0 & E'_{35} \\
E'_{13} & E'_{23} & E'_{33} & E'_{16} & -E'_{35} & 0 \\
\hline
\multicolumn{3}{c|}{\text{upper right, signs changed}} & \multicolumn{3}{c}{\text{upper left}}
\end{array}
\right\}, -\lambda<r\leq-\mu \\
\boldsymbol{O}, \text{ elsewhere.}
\end{array}
\right.
$$

$$
\begin{array}{c}
[\boldsymbol{E}'(\lambda)-\boldsymbol{E}'(\mu)]= \\
(0<\mu<\lambda)
\end{array}
\left\{
\begin{array}{l}
\left\{
\begin{array}{ccc|ccc}
E'_{11} & E'_{12} & E'_{13} & 0 & -E'_{24} & E'_{16} \\
E'_{12} & E'_{22} & E'_{23} & E'_{24} & 0 & -E'_{25} \\
E'_{13} & E'_{23} & E'_{33} & -E'_{16} & E'_{35} & 0 \\
\hline
\multicolumn{3}{c|}{\text{upper right, signs changed}} & \multicolumn{3}{c}{\text{upper left}}
\end{array}
\right\}, \mu<r\leq\lambda. \\
\boldsymbol{O}, \text{ elsewhere.}
\end{array}
\right.
$$

In addition to these, we shall have occasion to use the matrix $[\boldsymbol{E}'(0+0)-\boldsymbol{E}'(0-0)]$. Note that $\boldsymbol{E}'(0+0)$ can be had as a limiting case of $\boldsymbol{E}'(\lambda)$, $\lambda>0$, $\lambda\rightarrow 0$. Thus, $\boldsymbol{E}'(0+0)=\boldsymbol{E}'(\lambda)$ for $\lambda>0$, $r>\lambda$; and similarly, $\boldsymbol{E}'(0-0)=\boldsymbol{E}'(\lambda)$ for $\lambda<0$, $r>-\lambda$. The matrix in question can then easily be shown to be

$$
[\boldsymbol{E}'(0+0)-\boldsymbol{E}'(0-0)]=\frac{1}{r^2}
\left\{
\begin{array}{ccc|c}
x^2 & xy & xz & \\
yx & y^2 & yz & 0 \\
zx & zy & z^2 & \\
\hline
\multicolumn{3}{c|}{0} & \text{upper left}
\end{array}
\right\}, 0<r\leq+\infty.
$$

7. *Characteristic values and elements of* $\boldsymbol{H}'$ *and* $\boldsymbol{H}$. It is convenient at this point to say a few words regarding characteristic values and characteristic elements. For through a discussion of them, we are not only provided with a partial check on the calculation of $\boldsymbol{E}'(\lambda)$, but we shall in addition obtain a result which will be of use later on. A characteristic value of $\boldsymbol{H}'$ is defined [12] as a value of l for which $\boldsymbol{H}'_l$ has no inverse. Now the determinant of the matrix $\boldsymbol{H}'_l$ is $l^2(l^2-r^2)^2$; hence $l=0$ is the only characteristic value of $\boldsymbol{H}'$. $l=\pm r$ are not characteristic values, since for a fixed l, this equality holds only over the surface of a sphere—i. e., over a three-dimensional set of measure zero. A characteristic element of $\boldsymbol{H}'$ is defined as an element $\boldsymbol{g}\ \varepsilon\ L_{2,6}$,

[12] Stone, Definition 4. 2.

$\boldsymbol{g} \neq 0$, such that $\boldsymbol{H}'\boldsymbol{g} = l\boldsymbol{g}$, where l is a characteristic value. On the other hand, a necessary and sufficient condition [13] that $\boldsymbol{g}$ be a characteristic element of $\boldsymbol{H}'$ corresponding to the characteristic value l is that

$$[\boldsymbol{E}'(l+0) - \boldsymbol{E}'(l-0)]\boldsymbol{g} = \boldsymbol{g}.$$

If, then, the expressions obtained for $\boldsymbol{E}'(\lambda)$ are correct, the solutions of these last two equations for the characteristic value $l = 0$ must be the same.

The matrix equation $[\boldsymbol{E}'(0+0) - \boldsymbol{E}'(0-0)]\boldsymbol{g} = \boldsymbol{g}$ leads to six scalar equations, of which the first three are

$$\begin{aligned} (x^2 - r^2)g_1 + \quad & xyg_2 + \quad & xzg_3 &= 0 \\ yxg_1 + (y^2 & - r^2)g_2 + \quad & yzg_3 &= 0 \\ zxg_1 + \quad & zyg_2 + (z^2 - r^2)g_3 &&= 0, \end{aligned}$$

and the second three are different only in that g_1, g_2, g_3 are replaced by g_4, g_5, g_6 respectively. The two sets of three equations are independent; and in either case, for a fixed x, y, z, the determinant of the matrix of coefficients is zero, and the rank of the matrix is 2. Hence, in either case, any one of the three components involved may be chosen arbitrarily, and the rest will be uniquely determined. Put $g_1 = xp_1$, $g_4 = xp_2$, where p_1 and p_2 are to a certain extent arbitrary measurable functions of x, y, and z. Then

$$\boldsymbol{g} = (xp_1, yp_1, zp_1, xp_2, yp_2, zp_2).$$

Our choice of p_1 and p_2 is restricted in that the resulting vector $\boldsymbol{g}$ must belong to $L_{2,6}$. It is easy to verify that the solution of the equation $\boldsymbol{H}'\boldsymbol{g} = 0$ has the same form, as was to be expected.

We have just shown that a characteristic element of $\boldsymbol{H}'$ must be of the form $\boldsymbol{g} = (xp_1, yp_1, zp_1, xp_2, yp_2, zp_2)$, where p_1 and p_2 are any measurable functions such that $\boldsymbol{g} \in L_{2,6}$. Conversely, any such functions p_1 and p_2 will yield a $\boldsymbol{g}$ in the domain of $\boldsymbol{H}'$ such that $\boldsymbol{H}'\boldsymbol{g} = 0$. The set P' of all such functions $\boldsymbol{g}$ is a closed linear manifold, called the characteristic manifold of $\boldsymbol{H}'$. If the characteristic manifold of $\boldsymbol{H}$ be denoted by P, then, since the characteristic values of $\boldsymbol{H}$ and $\boldsymbol{H}'$ are identical [14] we must have $P \equiv \boldsymbol{T}^{-1}P'$. Suppose now that we restrict ourselves to a dense linear manifold P'_0 of P', and calculate $\boldsymbol{T}^{-1}P'_0$. We shall obtain a linear manifold P_0 which will be dense in P. This follows immediately from the fact that distances are preserved under the Fourier transformation.

[13] Stone, Theorem 5. 13.

[14] Stone, Theorem 4. 3.

Let us take as P'_0 the set of all functions $\boldsymbol{g} \varepsilon L_{2,6}$ of the form

$$\boldsymbol{g} = (xp_1, yp_1, zp_1, xp_2, yp_2, zp_2),$$

where p_1 and p_2 belong to L_2. It is not difficult to show that this set is dense in P'. If now we apply $\boldsymbol{T}^{-1}$ to $\boldsymbol{g}$, we obtain as P_0 the set of all functions of the form

$$\boldsymbol{f} = \left(\frac{\partial p_1}{\partial x}, \frac{\partial p_1}{\partial y}, \frac{\partial p_1}{\partial z}, \frac{\partial p_2}{\partial x}, \frac{\partial p_2}{\partial y}, \frac{\partial p_2}{\partial z}\right),$$

where p_1 and p_2 belong to L_2, are absolutely continuous in x, y, and z separately, and have the property that their first partial derivatives with respect to x, y, or z belong to L_2. This result follows directly from our work in Theorem IV.

8. *The operator* $\boldsymbol{E}(\lambda)$ *corresponding to* $\boldsymbol{H}$. It has already been pointed out in Section 2 that $\boldsymbol{Hf} = \boldsymbol{T}^{-1}\boldsymbol{H}'\boldsymbol{Tf}$ implies $\boldsymbol{E}(\lambda)\boldsymbol{f} = \boldsymbol{T}^{-1}\boldsymbol{E}'(\lambda)\boldsymbol{Tf}$. Since this relation holds for an arbitrary element $\boldsymbol{f} \varepsilon L_{2,6}$ we are entitled to write $E_{jk}(\lambda)f_k = T^{-1}E'_{jk}(\lambda)Tf_k$. A difficulty arises in the calculation of the $E_{jk}(\lambda)$ due to the fact that, in most instances, the $E'_{jk}(\lambda)$, as functions of x, y, z, do not belong to L_2, as we shall presently require. We shall find ultimately that it suits our purposes just as well to calculate

$$(8.1) \qquad [E_{jk}(\lambda) - E_{jk}(\mu)]f_k = T^{-1}[E'_{jk}(\lambda) - E'_{jk}(\mu)]Tf_k$$

for the cases $\mu < \lambda < 0$, and $0 < \mu < \lambda$. Put

$$[E_{jk}(\lambda) - E_{jk}(\mu)] \equiv E_{jk}(\Delta) \equiv E_{jk}(x, y, z; \Delta).$$

Then (8.1) becomes

$$(8.2) \quad E_{jk}(x, y, z; \Delta)f_k(x, y, z)$$
$$= \frac{1}{(2\pi)^{3/2}} \int\int\limits_{-\infty}^{+\infty}\int e^{-i(xx_2+yy_2+zz_2)} E'_{jk}(x_2, y_2, z_2; \Delta)\; Tf_k(x_2, y_2, z_2)\, dx_2 dy_2 dz_2.$$

Let us examine the right-hand member of this equation. It exists as a function of x, y, z, and belongs to L_2. To establish this fact, we argue as follows: since $f_k \varepsilon L_2$, so also does Tf_k. The effect of applying $E'_{jk}(\Delta)$ to any function is to multiply it by a bounded measurable function or by zero according as the point (x, y, z) lies inside or outside a region bounded by two concentric spherical surfaces. Hence $E'_{jk}(\Delta)Tf_k \varepsilon L_2$, and vanishes outside the above mentioned region. The right-hand member of (8.2) is nothing

but the inverse Fourier transformation applied to $E'_{jk}(\Delta)Tf_k$ and as such, it exists and belongs to L_2. Now let us look at the integrand from another point of view. Put

$$g_{jk}(x_2, y_2, z_2; x, y, z; \Delta) = e^{-i(xx_2+yy_2+zz_2)} E'_{jk}(x_2, y_2, z_2; \Delta).$$

Then g_{jk}, as a function of x_2, y_2, z_2, belongs to L_2. But it is well known that if f and $g \,\varepsilon\, L_2$, then

$$\iiint_{-\infty}^{+\infty} g \cdot Tf = \iiint_{-\infty}^{+\infty} Tg \cdot f,$$

both integrals being absolutely convergent. Thus, if we put for convenience $\xi = x_1 - x$, $\eta = y_1 - y$, $\zeta = z_1 - z$, we are entitled to write, instead of (8. 2), the following:

$$(8.\ 3)\quad \begin{aligned} &E_{jk}(x, y, z; \Delta) f_k(x, y, z) \\ &\qquad = \frac{1}{(2\pi)^{3/2}} \iiint_{-\infty}^{+\infty} G_{jk}(\xi, \eta, \zeta; \Delta) f_k(x_1, y_1, z_1)\, dx_1 dy_1 dz_1, \\ &G_{jk}(\xi, \eta, \zeta; \Delta) \\ &\qquad = \frac{1}{(2\pi)^{3/2}} \iiint_{-\infty}^{+\infty} e^{i(\xi x_2+\eta y_2+\zeta z_2)} E'_{jk}(x_2, y_2, z_2; \Delta)\, dx_2 dy_2 dz_2. \end{aligned}$$

We shall leave the $E_{jk}(\Delta)f_k$ in the integral form as above, and proceed to calculate the G_{jk}. Because of the fact that the $E'_{jk}(\Delta)$ vanish outside a region between two spherical surfaces, it is advisable to work with spherical rather than with rectangular coördinates. To this end, we put $x_2 = r\cos\theta\sin\phi$, $y_2 = r\sin\theta\sin\phi$, $z_2 = r\cos\phi$. We note next that the $E'_{jk}(\Delta)$ are made up of ten fractions, or combinations thereof. They are $1/2$; x^2, y^2, z^2, xy, xz, and yz, each divided by $2r^2$; and x, y, and z, each divided by $2r$. That is, if we put for convenience $\omega \equiv \xi\cos\theta\sin\phi + \eta\sin\theta\sin\phi + \zeta\cos\phi$, we must evaluate the following ten integrals:

$$I_1 = \int_\mu^\lambda \int_0^{2\pi} \int_0^\pi e^{ir\omega}\, r^2 \sin\phi\; dr d\theta d\phi$$

$$I_2 = \int_\mu^\lambda \int_0^{2\pi} \int_0^\pi e^{ir\omega}\, r^2 \sin^3\phi \cos^2\theta\; dr d\theta d\phi$$

$$I_3 = \int_\mu^\lambda \int_0^{2\pi} \int_0^\pi e^{ir\omega}\, r^2 \sin^3\phi \sin^2\theta\; dr d\theta d\phi$$

$$I_4 = \int_\mu^\lambda \int_0^{2\pi} \int_0^\pi e^{ir\omega}\, r^2 \sin\phi \cos^2\phi\; dr d\theta d\phi$$

$$I_5 = \int_\mu^\lambda \int_0^{2\pi} \int_0^\pi e^{ir\omega} r^2 \sin^3\phi \sin\theta \cos\theta \, drd\theta d\phi$$

$$I_6 = \int_\mu^\lambda \int_0^{2\pi} \int_0^\pi e^{ir\omega} r^2 \sin^2\phi \cos\phi \cos\theta \, drd\theta d\phi$$

$$I_7 = \int_\mu^\lambda \int_0^{2\pi} \int_0^\pi e^{ir\omega} r^2 \sin^2\phi \cos\phi \sin\theta \, drd\theta d\phi$$

$$I_8 = \int_\mu^\lambda \int_0^{2\pi} \int_0^\pi e^{ir\omega} r^2 \sin^2\phi \cos\theta \, drd\theta d\phi$$

$$I_9 = \int_\mu^\lambda \int_0^{2\pi} \int_0^\pi e^{ir\omega} r^2 \sin^2\phi \sin\theta \, drd\theta d\phi$$

$$I_{10} = \int_\mu^\lambda \int_0^{2\pi} \int_0^\pi e^{ir\omega} r^2 \sin\phi \cos\phi \, drd\theta d\phi.$$

These integrals, as written, are for the case $0 < \mu < \lambda$. Those for the case $\mu < \lambda < 0$ differ only in that the lower and upper limits for r are, respectively, $-\lambda$ and $-\mu$. We shall not go into any great detail regarding the calculation of these integrals, but it is perhaps advisable to note the various devices used. Let us first consider the integration with respect to θ of I_1. This involves

$$\int_0^{2\pi} e^{ir(b\sin\theta + c\cos\theta)} \, d\theta,$$

where we have put $b \equiv \eta \sin\phi$, $c \equiv \xi \sin\phi$. If now we let $\theta = \psi + \delta$ where δ is defined by the relation $\delta = \tan^{-1} b/c$, and if we put $d \equiv r(b^2 + c^2)^{1/2}$, the above integral becomes

$$\int_{-\delta}^{2\pi-\delta} e^{id\cos\psi} \, d\psi = \int_0^{2\pi} e^{id\cos\psi} \, d\psi = 2 \int_0^\pi e^{id\cos\psi} \, d\psi = 2\pi J_0(d),$$

where J_0 is the Bessel function of the first kind and 0-th order. Note that, because of our substitutions, $J_0(d) \equiv J_0(r\sqrt{\xi^2 + \eta^2} \sin\phi)$.

Similarly, I_2 involves

$$\int_0^{2\pi} e^{ir(b\sin\theta + c\cos\theta)} \cos^2\theta d\theta = -\int_0^{2\pi} \frac{1}{r^2} \frac{\partial^2}{\partial c^2} [e^{ir(b\sin\theta + c\cos\theta)}] \, d\theta$$

$$= -\frac{1}{r^2} \frac{\partial^2}{\partial c^2} \left[\int_0^{2\pi} e^{ir(b\sin\theta + c\cos\theta)} \, d\theta \right] = -\frac{2\pi}{r^2} \frac{\partial^2}{\partial c^2} [J_0(d)].$$

This last term can be broken down as follows:

$$\frac{\partial^2}{\partial c^2} [J_0(d)] = r^2 \frac{\xi^2}{\xi^2 + \eta^2} J_0''(d) + \frac{r}{\sin\phi} \frac{\eta^2}{(\xi^2 + \eta^2)^{3/2}} J'_0(d),$$

where the prime denotes differentiation with respect to the argument d. We have also $J_0'(d) = -J_1(d)$, $J_0''(d) = \frac{1}{2}[J_2(d) - J_0(d)]$. Hence I_2 can be

written as the sum of three integrals, involving J_0, J_1, and J_2 respectively. The integrals I_8 to I_{10} are treated in a similar manner.

The integration with respect to ϕ amounts to finding expressions for the following:

$$\text{(A)}\qquad \int_0^\pi e^{ir\xi\cos\phi} J_n(r\sqrt{\xi^2+\eta^2}\sin\phi)\sin^{n+1}\phi\, d\phi$$

$$\text{(B)}\qquad \int_0^\pi e^{ir\xi\cos\phi} J_n(r\sqrt{\xi^2+\eta^2}\sin\phi)\cos\phi\sin^{n+1}\phi\, d\phi$$

$$\text{(C)}\qquad \int_0^\pi e^{ir\xi\cos\phi} J_n(r\sqrt{\xi^2+\eta^2}\sin\phi)\cos^2\phi\sin^{n+1}\phi\, d\phi.$$

The following formula, which has been derived by both N. Sonine and N. Nielsen,[15] is of basic importance in this respect:

$$\text{(S)}\qquad \int_0^{\pi/2} J_m(aq\cos\phi)J_n(az\sin\phi)\cos^{m+1}\phi\sin^{n+1}\phi\, d\phi = \frac{q^m z^n}{a}\,\frac{J_{m+n+1}(a\sqrt{q^2+z^2})}{(q^2+z^2)^{(m+n+1)/2}}.$$

(In the general case, where m and n may be complex, the only restriction placed upon them is that their real parts be greater than -1). Sonine showed in addition that if we put $m = -1/2$, and make use of the relation $J_{-1/2}(t) = (2/\pi t)^{1/2}\cos t$, we obtain

$$\int_0^{\pi/2}\cos(aq\cos\phi)J_n(az\sin\phi)\sin^{n+1}\phi\, d\phi = \tfrac{1}{2}\int_0^\pi \cos(aq\cos\phi)J_n(az\sin\phi)\sin^{n+1}\phi\, d\phi = \left(\frac{\pi}{2a}\right)^{1/2} z^n\,\frac{J_{n+1/2}(a\rho)}{\rho^{n+1/2}},$$

where $\rho \equiv (q^2+z^2)^{1/2}$. Finally, if we note that the value of the last integral is zero if we replace $\cos(aq\cos\phi)$ by $\sin(aq\cos\phi)$, we have immediately an expression for (A), to wit:

$$\text{(A}'\text{)}\qquad \int_0^\pi e^{iaq\cos\phi}J_n(az\sin\phi)\sin^{n+1}\phi\, d\phi = \left(\frac{2\pi}{a}\right)^{1/2} z^n\,\frac{J_{n+1/2}(a\rho)}{\rho^{n+1/2}}.$$

It will be found that we can evaluate (B) and (C) by putting $m = 1/2$ and $m = 3/2$ respectively in (S), noting that $J_{1/2}(t) = (2/\pi t)^{1/2}\sin t$, $J_{3/2}(t) = (1/t)J_{1/2}(t) - J_{-1/2}(t)$. We obtain

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[15] N. Sonine, *Mathematische Annalen*, vol. 16 (1880), p. 36; N. Nielsen, *Handbuch der Cylinderfunktionen* (1904), p. 181.

(B′) $$\int_0^\pi e^{iaq\cos\phi} J_n(az\sin\phi)\cos\phi\sin^{n+1}\phi\, d\phi = i\left(\frac{2\pi}{a}\right)^{1/2} qz^n \frac{J_{n+3/2}(a\rho)}{\rho^{n+3/2}}$$

(C′) $$\int_0^\pi e^{iaq\cos\phi} J_n(az\sin\phi)\cos^2\phi\sin^{n+1}\phi\, d\phi$$
$$= \left(\frac{2\pi}{a}\right)^{1/2} z^n \left[\frac{1}{a}\,\frac{J_{n+3/2}(a\rho)}{\rho^{n+3/2}} - q^2\frac{J_{n+5/2}(a\rho)}{\rho^{n+5/2}}\right].$$

To adapt these formulae to our use, we have only to put r for a, ζ for q, and $(\xi^2+\eta^2)^{1/2}$ for z. Then $\rho \equiv (q^2+z^2)^{1/2}$ becomes $(\xi^2+\eta^2+\zeta^2)^{1/2}$. We shall continue to designate this last radical by ρ, thereby conforming to our use of r for $(x^2+y^2+z^2)^{1/2}$.

The integration with respect to r presents no particular difficulty. Bessel's functions of low orders which are half of odd integers are readily reduced to trigonometric functions. Once the ten integrals are evaluated, the G_{jk} are found to be as follows:

$$G_{11} = \frac{1}{(2\pi)^{1/2}\rho^5}\Big[2(-\xi^2+\eta^2+\zeta^2)(\sin\rho\lambda - \sin\rho\mu)$$
$$-(\eta^2+\zeta^2)(\rho\lambda\cos\rho\lambda - \rho\mu\cos\rho\mu) + (2\xi^2-\eta^2-\zeta^2)\int_{\rho\mu}^{\rho\lambda}\frac{\sin u}{u}\,du\Big]$$

$G_{22} = G_{11}$ with ξ and η interchanged.
$G_{33} = G_{11}$ with ξ and ζ interchanged.

$$G_{12} = \frac{-\xi\eta}{(2\pi)^{1/2}\rho^5}\Big[4(\sin\rho\lambda - \sin\rho\mu)$$
$$-(\rho\lambda\cos\rho\lambda - \rho\mu\cos\rho\mu) - 3\int_{\rho\mu}^{\rho\lambda}\frac{\sin u}{u}\,du\Big]$$

$G_{13} = G_{12}$ with η and ζ interchanged.
$G_{23} = G_{12}$ with ξ and ζ interchanged.

$$G_{35} = \frac{-\xi}{i(2\pi)^{1/2}\rho^4}\left[2(\cos\rho\lambda - \cos\rho\mu) + (\rho\lambda\sin\rho\lambda - \rho\mu\sin\rho\mu)\right]$$

$G_{16} = G_{35}$ with ξ and η interchanged.
$G_{24} = G_{35}$ with ξ and ζ interchanged.
The matrix $\boldsymbol{G}$ is given by

$$\underset{\begin{pmatrix}\mu<\lambda<0\\ 0<\mu<\lambda\end{pmatrix}}{\boldsymbol{G}(\xi,\eta,\zeta;\Delta)} = \left\{\begin{array}{ccc|ccc} G_{11} & G_{12} & G_{13} & 0 & -G_{24} & G_{16} \\ G_{12} & G_{22} & G_{23} & G_{24} & 0 & -G_{35} \\ G_{13} & G_{23} & G_{33} & -G_{16} & G_{35} & 0 \\ \hline & \text{upper right,} & & & \text{upper} & \\ & \text{signs changed} & & & \text{left} & \end{array}\right\}.$$

In connection with these results, we note the important fact that the matrix G is the same for the two cases $\mu < \lambda < 0$ and $0 < \mu < \lambda$. Also, we are afforded a partial check on our calculations in that since the $E'_{jk}(\Delta) \varepsilon L_2$, so also should the $G_{jk}(\Delta)$, as functions of ξ, η, ζ, have this property. That they actually do is easy to verify.

9. *The operator* $\boldsymbol{F}(\boldsymbol{H}) \equiv e^{itH}$. We are now prepared to show that the operator $\boldsymbol{F}(\boldsymbol{H}) \equiv e^{itH}$ is a matrix whose components $F_{jk}(\boldsymbol{H})$ are given by the relations

$$(9.1) \quad F_{jk}(\boldsymbol{H}) f_k(x, y, z)$$
$$= \underset{A\to\infty}{\mathrm{l.i.m.}} \frac{1}{(2\pi)^{3/2}} \iiint\limits_{-\infty}^{+\infty} D_{jk}(\xi, \eta, \zeta; t, A) f_k(x_1, y_1, z_1)\, dx_1 dy_1 dz_1,$$
$$D_{jk}(\xi, \eta, \zeta; t, A) \equiv \left[\int_{-A}^{0-0} + \int_{0+0}^{A} \right] e^{it\lambda}\, d_\lambda G_{jk}(\xi, \eta, \zeta; \Delta),$$

wherein the G_{jk} are the elements of the matrix determined in the last section. Before setting out to prove this relation, however, we shall make a few observations regarding the calculation of the D_{jk}.

It was found in the last section that there were only nine distinct G_{jk} (with the exception of a sign), which in turn could be classified as three groups of three each, those elements in each group differing from each other only insofar as ξ, η, ζ are concerned. Hence for the moment we need consider only one element in each group—say G_{11}, G_{12}, G_{35}. We have

$$d_\lambda G_{11}(\Delta) = \frac{1}{(2\pi)^{1/2}\rho^4} \Big[(-2\xi^2 + \eta^2 + \zeta^2) \cos \rho\lambda + (\eta^2 + \zeta^2)\rho\lambda \sin \rho\lambda$$
$$+ (2\xi^2 - \eta^2 - \zeta^2) \frac{\sin \rho\lambda}{\rho\lambda} \Big]\, d\lambda$$

$$d_\lambda G_{12}(\Delta) = \frac{-\xi\eta}{(2\pi)^{1/2}\rho^4} \left[3 \cos \rho\lambda + \rho\lambda \sin \rho\lambda - 3 \frac{\sin \rho\lambda}{\rho\lambda} \right] d\lambda$$

$$d_\lambda G_{35}(\Delta) = \frac{-\xi}{i(2\pi)^{1/2}\rho^3} [\rho\lambda \cos \rho\lambda - \sin \rho\lambda]\, d\lambda.$$

Now all the terms involved above, multiplied by $e^{it\lambda}$, are continuous at the origin. Hence, we may replace $\left[\int_{-A}^{0-0} + \int_{0+0}^{A} \right]$ by $\int_{-A}^{A}$. The calculation of the D_{jk} is a relatively simple matter. We shall not list the results here, as it seems advisable, for all purposes of reference, to include them in the summary (Section 11). It is a matter of simple routine to show that, as functions of x_1, y_1, z_1, the $D_{jk} \varepsilon L_2$.

We have now to verify the relation (9.1). Among the important characteristic properties of a resolution of the identity are the following: [16]

$$\underset{\lambda\to+\infty}{\text{l.i.m.}}\, \boldsymbol{E}(\lambda)\boldsymbol{f} = \boldsymbol{f}, \qquad \underset{\lambda\to-\infty}{\text{l.i.m.}}\, \boldsymbol{E}(\lambda)\boldsymbol{f} = 0$$

for any function $\boldsymbol{f}$ in its domain. In the case at hand, then,

$$\underset{A\to\infty}{\text{l.i.m.}}\, [\boldsymbol{E}(A) - \boldsymbol{E}(-\Lambda)]\boldsymbol{f} = \boldsymbol{f}.$$

Now by Stone, Theorem 6.6, $e^{it\boldsymbol{H}}$ is a unitary transformation.[17] Hence we must have

$$e^{it\boldsymbol{H}}\boldsymbol{f} = \underset{A\to\infty}{\text{l.i.m.}}\, e^{it\boldsymbol{H}}[\boldsymbol{E}(A) - \boldsymbol{E}(-A)]\boldsymbol{f}, \tag{9.2}$$

for every $\boldsymbol{f} \,\varepsilon\, L_{2,6}$. Consider now

$$[\boldsymbol{E}(A) - \boldsymbol{E}(-A)]\boldsymbol{f} = \{[\boldsymbol{E}(A) - \boldsymbol{E}(0+0)] - [\boldsymbol{E}(-A) - \boldsymbol{E}(0-0)]\}\boldsymbol{f} + [\boldsymbol{E}(0+0) - \boldsymbol{E}(0-0)]\boldsymbol{f}, \qquad A > 0.$$

Regarding the last term on the right-hand side of this relation, it was pointed out in Section 7 that the range of the operator $[\boldsymbol{E}(0+0) - \boldsymbol{E}(0-0)]$ consists of those and only those functions which are characteristic elements of $\boldsymbol{H}$ corresponding to the characteristic value $l = 0$. The term in question can thus be eliminated from our present consideration. For if $\boldsymbol{f}$ is a characteristic element, then $\boldsymbol{H}\boldsymbol{f} = 0$. Referring back to our original problem—that of determining the solution of the equation $\partial\boldsymbol{f}/\partial t = i\boldsymbol{H}\boldsymbol{f}$—it is evident that in such a case $\partial\boldsymbol{f}/\partial t = 0$, and the function $\boldsymbol{f}$ is constant in time. Such functions we can afford to disregard in the present analysis.[18] It should be pointed out, however, that we do not eliminate the term $[\boldsymbol{E}(0+0) - \boldsymbol{E}(0-0)]\boldsymbol{f}$ as a matter of mere convenience, but because of the serious difficulties it would cause in the work which is to follow. We shall hence forth understand, when we write $\boldsymbol{E}(A)$ and $\boldsymbol{E}(-A)$, that we mean $[\boldsymbol{E}(A) - \boldsymbol{E}(0+0)]$ and $[\boldsymbol{E}(-\Lambda) - \boldsymbol{E}(0-0)]$ respectively.

Now let $\phi_n(t,\lambda)$ be a function which for t fixed converges uniformly to $e^{it\lambda}$ on $(-A, A)$. Then in accordance with Stone, Theorem 6.1,

$$e^{it\boldsymbol{H}}[\boldsymbol{E}(A) - \boldsymbol{E}(-A)]\boldsymbol{f} = \underset{n\to\infty}{\text{l.i.m.}}\, \phi_n(t,\lambda)[\boldsymbol{E}(A) - \boldsymbol{E}(-A)]\boldsymbol{f}. \tag{9.3}$$

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[16] Stone, Definition 5.1.

[17] The reader should note the revision of notation brought about in Stone, Definition 6.4. In accordance with this revision, we are using $\boldsymbol{F}(\boldsymbol{H})$ in the place of $\boldsymbol{T}(\boldsymbol{F})$ in Theorem 6.1.

[18] See also Section 10.

We choose $\phi_n(t,\lambda)$ as the step-function defined as follows. Divide the interval $(-A, A)$ into n parts, the only restriction being that the origin be interior to no sub-interval. This is, of course, no real restriction, since we could just as well work with the two intervals $(-A, 0-0)$ and $(0+0, A)$ separately. Put

$$\phi_n(t,\lambda) = \begin{cases} e^{i\lambda_\nu t}, & \lambda_\nu \leq \lambda \leq \lambda_{\nu+1} \\ 0, & \text{elsewhere.} \end{cases} \qquad (\nu = 1, \cdots, n),$$

It is apparent that $\phi_n(t,\lambda)$ has the property which we required of it above. Put for simplicity

$$\boldsymbol{W}(\boldsymbol{H}; A) \equiv e^{it\boldsymbol{H}}[\boldsymbol{E}(A) - \boldsymbol{E}(-A)],$$

and denote as usual its elements by $W_{jk}(\boldsymbol{H}; A)$. Since the relation (9.3) holds for an arbitrary $f \varepsilon L_{2,6}$, we must have, for the separate components,

$$W_{jk}(\boldsymbol{H}; A) f_k = \underset{n\to\infty}{\text{l.i.m.}} \left\{ \sum_{\nu=0}^{n} e^{i\lambda_\nu t}[E_{jk}(\lambda_{\nu+1}) - E_{jk}(\lambda_\nu)] \right\} f_k.$$

But $E_{jk}(\lambda_{\nu+1}) - E_{jk}(\lambda_\nu)$ is nothing other than $E_{jk}(\Delta)$ where $\lambda = \lambda_{\nu+1}$, $\mu = \lambda_\nu$. If we denote this by $E_{jk}(\Delta_\nu)$, we have in view of (9.3),

$$(9.4) \quad W_{jk}(\boldsymbol{H}; A) f_k = \underset{n\to\infty}{\text{l.i.m.}} \frac{1}{(2\pi)^{3/2}} \int\int\int_{-\infty}^{+\infty} \sum_{\nu=0}^{n} {}^{i\lambda_\nu t} G_{jk}(\xi,\eta,\zeta;\Delta_\nu) \cdot f_k(x_1,y_1,z_1)\, dx_1 dy_1 dz_1$$

$$= \underset{n\to\infty}{\text{l.i.m.}} \frac{1}{(2\pi)^{3/2}} \int\int\int_{-\infty}^{+\infty} \left\{ \left[\int_{-A}^{0-0} + \int_{0+0}^{A} \right] \phi_n(t,\lambda) \cdot d_\lambda G_{jk}(\xi,\eta,\zeta;\Delta) \right\} f_k(x_1,y_1,z_1)\, dx_1 dy_1 dz_1.$$

It remains to show that (9.4) implies

$$(9.5) \quad W_{jk}(\boldsymbol{H}; A) f_k = \frac{1}{(2\pi)^{3/2}} \int\int\int_{-\infty}^{+\infty} \left\{ \left[\int_{-A}^{0-0} + \int_{0+0}^{A} \right] e^{it\lambda} \cdot d_\lambda G_{jk}(\xi,\eta,\zeta;\Delta) \right\} f_k(x_1,y_1,z_1)\, dx_1 dy_1 dz_1$$

almost everywhere.

We introduce the functions $f_k{}^B$ which are identically equal to f_k inside and on an arbitrary origin-centered, axis-parallel cube of side $2B$, and zero elsewhere. We then consider the integral

$$(9.6) \qquad \int\int_{-\infty}^{+\infty}\int \Big\{ \Big[\int_{-A}^{0-0} + \int_{0+0}^{A} \Big] [e^{it\lambda} - \phi_n(t,\lambda)] d_\lambda G_{jk}(\xi,\eta,\zeta;\Delta) \Big\} \cdot f_k^B(x_1,y_1,z_1)\, dx_1 dy_1 dz_1.$$

Since for t fixed, $\phi_n(t,\lambda)$ converges uniformly to $e^{it\lambda}$ on $(-A, A)$, there exists a sequence of constants $\{M_n\}$ depending only on n, such that (i) $| e^{it\lambda} - \phi_n(t,\lambda) | \leq M_n$ for every n, and (ii) $\lim_{n\to\infty} M_n = 0$. Hence

$$\Big| \Big[\int_{-A}^{0-0} + \int_{0+0}^{A} \Big] [e^{it\lambda} - \phi_n(t,\lambda)] d_\lambda G_{jk}(\xi,\eta,\zeta;\Delta) \Big|$$
$$\leq M_n V_{-A}^{A}(G_{jk}) \leq M_n \Big[\int_{-A}^{0-0} + \int_{0+0}^{A} \Big] | G'_{jk}(\xi,\eta,\zeta;\Delta) |\, d\lambda,$$

where the prime denotes differentiation with respect to λ. It is easy to show that the integral on the right-hand side of this relation exists as a function of x_1, y_1, z_1, and is of integrable square over the cube of side $2B$. If we denote the integral in question by $N_{jk}(\xi,\eta,\zeta;A)$, then for fixed x, y, z, the integral (9.6) exists and is dominated by

$$M_n \int\int_{-\infty}^{+\infty}\int N_{jk}(\xi,\eta,\zeta;A)\, | f_k^B(x_1,y_1,z_1) |\, dx_1 dy_1 dz_1.$$

This observation, together with the fact that $M_n \to 0$ as $n \to \infty$ insures the convergence of (9.6) to zero as $n \to \infty$. In that case,

$$(9.7) \qquad \frac{1}{(2\pi)^{3/2}} \int\int_{-\infty}^{+\infty}\int \Big\{ \Big[\int_{-A}^{0-0} + \int_{0+0}^{A} \Big] e^{it\lambda} \cdot d_\lambda G_{jk}(\xi,\eta,\zeta;\Delta) \Big\} f_k^B(x_1,y_1,z_1)\, dx_1 dy_1 dz_1$$
$$= \lim_{n\to\infty} \frac{1}{(2\pi)^{3/2}} \int\int_{-\infty}^{+\infty}\int \Big\{ \Big[\int_{-A}^{0-0} + \int_{0+0}^{A} \Big] \phi_n(t,\lambda) \cdot d_\lambda G_{jk}(\xi,\eta,\zeta;\Delta) \Big\} f_k^B(x_1,y_1,z_1)\, dx_1 dy_1 dz_1$$

for all x, y, z. Now the terms under l.i.m. and lim in (9.4) and (9.7) respectively are identical. Hence, by a well known theorem regarding the relation between limits in the mean and ordinary limits, the left-hand sides of these equations must be equal almost everywhere. That is, the relation (9.5) is true for all functions f_k^B. Let us write this in the form

(9.8) $W_{jk}(\boldsymbol{H}; A)f_k^B$

$$= \frac{1}{(2\pi)^{3/2}} \int\int\int_{-\infty}^{+\infty} D_{jk}(\xi, \eta, \zeta; t, A) f_k^B(x_1, y_1, z_1)\, dx_1 dy_1 dz_1,$$

$$D_{jk}(\xi, \eta, \zeta; t, A) \equiv \left[\int_{-A}^{0-0} + \int_{0+0}^{A} \right] e^{it\lambda} d_\lambda G_{jk}(\xi, \eta, \zeta; \Delta),$$

almost everywhere.

We now remove the restrictions on f_k. To do so, we note that $f_k = \underset{B\to\infty}{\text{l.i.m.}}\, f_k^B$. It is then not difficult to show that $W_{jk}(\boldsymbol{H}; A)f_k = \underset{B\to\infty}{\text{l.i.m.}}\, W_{jk}(\boldsymbol{H}; A)f_k^B$. In addition, since the $D_{jk} \in L_2$ as functions of x_1, y_1, z_1, we must have, for fixed x, y, z,

$$\frac{1}{(2\pi)^{3/2}} \int\int\int_{-\infty}^{+\infty} D_{jk} f_k\, dx_1 dy_1 dz_1 = \lim_{B\to\infty} \int\int\int_{-\infty}^{+\infty} D_{jk} f_k^B\, dx_1 dy_1 dz_1.$$

By the same argument just used regarding l.i.m. and lim, the relation (9.8) must hold for all functions $f_k \in L_2$. This, together with (9.2), completes the proof of (9.1).

10. *The equations* $\nabla \cdot \boldsymbol{e} = 0$ *and* $\nabla \cdot \boldsymbol{h} = 0$. Recall that of the four electrodynamic field equations, we have so far failed to consider the two equations $\nabla \cdot \boldsymbol{e} = 0$ and $\nabla \cdot \boldsymbol{h} = 0$. We propose in this section to investigate their significance.

We discussed in Section 7 a linear manifold dense in P, the characteristic manifold of $\boldsymbol{H}$. This was the set P_0, consisting of all functions $\boldsymbol{f}$ of the form

$$\boldsymbol{f} = \left(\frac{\partial p_1}{\partial x}, \frac{\partial p_1}{\partial y}, \frac{\partial p_1}{\partial z}, \frac{\partial p_2}{\partial x}, \frac{\partial p_2}{\partial y}, \frac{\partial p_2}{\partial z} \right),$$

where p_1 and p_2 belong to L_2, are absolutely continuous in x, y, and z separately, and have the property that their first partial derivatives with respect to x, y, or z belong to L_2. Let us examine the relation between the orthogonal complements of P and P_0. If we denote these manifolds by Q and Q_0 respectively, it is apparent that, since $P_0 \subset P$, $Q_0 \supseteq Q$. To show that $Q_0 = Q$, we argue as follows:

Since any function in $L_{2,6}$ can be uniquely represented as the sum of two functions, one in P and the other in Q, we have only to show that no function of $P - P_0$ is orthogonal to every function of P_0. Suppose, then, that $\boldsymbol{f} \in P - P_0$. Then a sequence of functions $\{\boldsymbol{f}_n\} \in P_0$ can be found such that $|\boldsymbol{f}_n - \boldsymbol{f}|^2 \equiv |\boldsymbol{f}_n|^2 + |\boldsymbol{f}|^2 - (\boldsymbol{f}, \boldsymbol{f}_n) - (\boldsymbol{f}_n, \boldsymbol{f}) \to 0$ as $n \to \infty$. Now assume $\boldsymbol{f}$ orthogonal to every function of P_0. Then the last relation becomes

$|f_n|^2 + |f|^2 \to 0$ as $n \to \infty$. This can be true only for $f \equiv 0$. But the null element belongs to Q as well as to P. Hence the orthogonal complement of P_0 coincides with that of P.

The set $Q_0 = Q$ must consist of all functions $g \in L_{2,6}$ such that

$$(10.1) \qquad \int\int\int_{-\infty}^{+\infty} g \cdot \bar{f}\, dxdydz = \lim_{A\to\infty} \int\int\int_{-A}^{A} g \cdot \bar{f}\, dxdydz = 0$$

for every $f \in P_0$. If the second term of this relation be integrated by parts, we obtain

$$(10.2) \qquad \begin{aligned} \lim_{A\to\infty} \Big\{ & \int\int_{-A}^{A} [g_1\bar{p}_1]_{x=-A}^{x=A} dydz + \int\int_{-A}^{A} [g_2\bar{p}_1]_{y=-A}^{y=A} dxdz + \int\int_{-A}^{A} [g_3\bar{p}_1]_{z=-A}^{z=A} dxdy \\ & + 3 \text{ similar terms involving the products of } \bar{p}_2 \text{ and } g_4, g_5, g_6, \text{ respectively} \\ & - \int\int\int_{-A}^{A} \Big[\Big(\frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} + \frac{\partial g_3}{\partial z}\Big)\bar{p}_1 + \Big(\frac{\partial g_4}{\partial x} + \frac{\partial g_5}{\partial y} + \frac{\partial g_6}{\partial z}\Big)\bar{p}_2\Big] dxdydz \Big\} = 0. \end{aligned}$$

It is now possible to pick out a sequence $\{a\}$ of A's such that the first six terms of this expression vanish in the limit. To show this, we argue as follows.

Consider any one of the terms in question—say the first. Since g_1 and $\bar{p}_1$ both belong to L_2, their product, which we shall denote by q, belongs to L_1. Then for a given ϵ, we can find a b so large that

$$\int_b^\infty \Big[\int\int_{-\infty}^{+\infty} |q(x,y,z)|\, dydz\Big] dx < \epsilon^2;$$

in which case, a fortiori,

$$\int_b^\infty \phi(x,c)\, dx \equiv \int_b^\infty \Big[\int\int_{-c}^{c} |q(x,y,z)|\, dydz\Big] dx < \epsilon^2.$$

It follows that the set of values of x, $x > b$, for which $\phi(x,c) \geqq \epsilon$ is of measure $< \epsilon$. That is, outside a set of values of x, $b < x \leq \infty$, of measure $< \epsilon$, $\phi(x,c) < \epsilon$. If now we take $c > b$, we can choose an a, $b < a < c$, so that $\phi(a,c) < \epsilon$. It follows that

$$\Big|\int\int_{-a}^{a} q(a,y,z)\, dydz\Big| \leq \int\int_{-a}^{a} |q(a,y,z)|\, dydz < \epsilon,$$

and as $\epsilon \to 0$ over any sequence of values, we can find corresponding values of $a \to \infty$. Now the first term of (10.2) is of the form

$$\lim_{A\to\infty} \left\{ \int\int_{-A}^{A} q(A, y, z)\, dydz - \int\int_{-A}^{A} q(-A, y, z)\, dydz \right\}.$$

It is evident not only that the sequence $\{a\}$ of A's can be so chosen that both parts of this first term vanish simultaneously in the limit, but that it can be so chosen that the same will be true of both parts of each of the first six terms of (10.2).

The condition (10.1) is consequently equivalent to

$$\int\int_{-\infty}^{+\infty}\int \left[\left(\frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} + \frac{\partial g_3}{\partial z}\right) \bar{p}_1 + \left(\frac{\partial g_4}{\partial x} + \frac{\partial g_5}{\partial y} + \frac{\partial g_6}{\partial z}\right) \bar{p}_2\right] dxdydz = 0.$$

This in turn is equivalent to

$$\frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} + \frac{\partial g_3}{\partial z} = 0, \qquad \frac{\partial g_4}{\partial x} + \frac{\partial g_5}{\partial y} + \frac{\partial g_6}{\partial z} = 0$$

almost everywhere; or if $\boldsymbol{g} \equiv (e_x, e_y, e_z, h_x, h_y, h_z)$, equivalent to the conditions $\nabla \cdot \boldsymbol{e} = 0$, $\nabla \cdot \boldsymbol{h} = 0$ almost everywhere.

We are thus led to the conclusion that of all functions belonging to $L_{2,6}$, those and only those which satisfy the divergence conditions constitute the orthogonal complement of the characteristic manifold of $\boldsymbol{H}$.

11. *Summary.* The object of this paper has been to obtain rigorously a general integral representation for the solution of the classical electrodynamic field equations in the case of empty space containing no charges or currents. These equations were expressed in the form $\nabla \cdot \boldsymbol{e} = 0$, $\nabla \cdot \boldsymbol{h} = 0$, $\nabla \times \boldsymbol{e} = -\partial \boldsymbol{h}/\partial t$, $\nabla \times \boldsymbol{h} = \partial \boldsymbol{e}/\partial t$. The significance of the two divergence equations was discussed in the last section. It was found that upon putting $\boldsymbol{v} \equiv (e_x, e_y, e_z, e_x, h_y, h_z)$, the two curl equations could be written as the single matrix equation

(11.1) $$\partial \boldsymbol{v}/\partial t = i\boldsymbol{H}\boldsymbol{v},$$

$$
\boldsymbol{H} \equiv \left\{ \begin{array}{c|ccc}
 & 0 & -\frac{i}{1}\frac{\partial z}{\partial} & \frac{i}{1}\frac{\partial y}{\partial} \\
\mathbf{0} & \frac{1}{i}\frac{\partial}{\partial z} & 0 & -\frac{1}{i}\frac{\partial}{\partial x} \\
 & -\frac{1}{i}\frac{\partial}{\partial y} & \frac{1}{i}\frac{\partial}{\partial x} & 0 \\
\hline
\begin{array}{c}\text{upper right,}\\ \text{signs changed}\end{array} & & \mathbf{0} &
\end{array} \right\}.
$$

The domain of the operator $\boldsymbol{H}$ was not precisely determined, but it is sufficient for all practical purposes to know that it contains the class of all vector functions $\boldsymbol{v}$ whose components along with their first partial derivatives with respect to x, y, or z belong to L_2 over all space, and whose components in addition are absolutely continuous in x, y, and z separately.

The solution of the equation (11. 1) was found to be

$$
\begin{aligned}
\boldsymbol{v} &= e^{it\boldsymbol{H}}\boldsymbol{v}_0 \\
&= \underset{A\to\infty}{\text{l.i.m.}}\, \frac{1}{(2\pi)^{3/2}} \int\int\limits_{-A}^{A}\int \boldsymbol{D}(\xi,\eta,\zeta;t,A)\,\boldsymbol{v}_0(x_1,y_1,z_1)\,dx_1dy_1dz_1,
\end{aligned}
$$

where $\xi \equiv x_1 - x$, $\eta \equiv y_1 - y$, $\zeta \equiv z_1 - z$, and where

$$
\boldsymbol{D}(\xi,\eta,\zeta;t,A) \equiv \left\{ \begin{array}{ccc|ccc}
D_{11} & D_{12} & D_{13} & 0 & -D_{24} & D_{16} \\
D_{12} & D_{22} & D_{23} & D_{24} & 0 & -D_{35} \\
D_{13} & D_{23} & D_{33} & -D_{16} & D_{35} & 0 \\
\hline
 & \begin{array}{c}\text{upper right,}\\ \text{signs changed}\end{array} & & & \begin{array}{c}\text{upper}\\ \text{left}\end{array} &
\end{array} \right\}.
$$

The elements of this matrix are given by the following expressions, wherein $\rho \equiv (\xi^2 + \eta^2 + \zeta^2)^{1/2}$.

$$
\begin{aligned}
D_{11} = \frac{1}{(2\pi)^{1/2}\rho^4} \Bigg\{ & \frac{t(-2\xi^2+\eta^2+\zeta^2) + 2\rho(-\xi^2+\eta^2+\zeta^2)}{(t+\rho)^2} \sin(t+\rho)A \\
& + \frac{t(-2\xi^2+\eta^2+\zeta^2) - 2\rho(-\xi^2+\eta^2+\zeta^2)}{(t-\rho)^2} \sin(t-\rho)A \\
& - A\rho(\eta^2+\zeta^2)\left[\frac{\cos(t+\rho)A}{t+\rho} - \frac{\cos(t-\rho)A}{t-\rho}\right] \\
& + \frac{2\xi^2-\eta^2-\zeta^2}{\rho}\int_{-A}^{A} e^{it\lambda}\frac{\sin\rho\lambda}{\lambda}\,d\lambda \Bigg\}.
\end{aligned}
$$

$D_{22} = D_{11}$ with ξ and η interchanged.
$D_{33} = D_{11}$ with ξ and ζ interchanged.

$$D_{12} = \frac{-\xi\eta}{(2\pi)^{1/2}\rho^4}\left\{\frac{3t+4\rho}{(t+\rho)^2}\sin(t+\rho)A + \frac{3t-4\rho}{(t-\rho)^2}\sin(t-\rho)A \right.$$
$$\left. -A\rho\left[\frac{\cos(t+\rho)A}{t+\rho} - \frac{\cos(t-\rho)A}{t-\rho}\right] - \frac{3}{\rho}\int_{-A}^{A} e^{it\lambda}\frac{\sin\rho\lambda}{\lambda}\,d\lambda\right\}$$

$D_{13} = D_{12}$ with η and ζ interchanged.
$D_{23} = D_{12}$ with ξ and ζ interchanged.

$$D_{35} = \frac{-\xi}{(2\pi)^{1/2}\rho^3}\left\{\frac{t+2\rho}{(t+\rho)^2}\sin(t+\rho)A - \frac{t-2\rho}{(t-\rho)^2}\sin(t-\rho)A \right.$$
$$\left. -A\rho\left[\frac{\cos(t+\rho)A}{t+\rho} + \frac{\cos(t-\rho)A}{t-\rho}\right]\right\}$$

$D_{16} = D_{35}$ with ξ and η interchanged.
$D_{24} = D_{35}$ with ξ and ζ interchanged.

Finally, if the results are desired in Heaviside-Lorentz units, we have only to replace t by ct, where c is the velocity of light *in vacuo*.

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CURVATURE TRAJECTORIES.[1]

By George Comenetz.

1. The geometrical objects which we study in this paper are triply infinite families of plane curves. We deal with three interesting special types of such families: the *dynamical, sectional,* and *curvature* types. These are related, in that all of their families possess the differential property which Kasner has given as dynamical property *I* (quoted below). Moreover, the body of families belonging to each one of the types is invariant under the group of all projective point transformations. A first set of three problems is suggested by the common differential property: to determine what families of curves appear under some two of the three types at once. A second set of three problems may be derived from the projective character: to determine which families under each type are entirely composed of conic sections. For two questions in each set the solutions are already known. The remaining pair of questions, connected with the curvature type, are answered below. A remarkable feature of this group of problems is that in every case an explicit answer can be obtained.

I wish to thank Professor Kasner for setting these problems, and for his assistance in solving them.

2. First we define the three types of families in question.

With an arbitrarily given positional field of force there is associated a family of *dynamical trajectories.* These are the totality of curves along which a mass particle can move under the influence of the field. Since the particle can start from any point, in any direction, and with any velocity, it can traverse ∞^3 different paths. The differential equation of these ∞^3 curves is

$$(1)\qquad y''' = \frac{\psi_x + (\psi_y - \phi_x)y' - \phi_y y'^2}{\psi - y'\phi}\, y'' - \frac{3\phi}{\psi - y'\phi}\, y''^2,$$

where $\phi(x, y)$ and $\psi(x, y)$ are the components of force.[2]

The most familiar example of a dynamical family is the set of all conics with a common focus, the orbits in the Newtonian field.

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[1] Abstract in *Bulletin of the American Mathematical Society,* March 1934, p. 213.

[2] E. Kasner, "The trajectories of dynamics," *Transactions of the American Mathematical Society,* vol. 7 (1906), pp. 401-424. Also "Differential-geometric aspects of dynamics," *Princeton Colloquium Lectures on Mathematics* (1913; new edition 1934).

The projective character of dynamical families was discovered by Appell.[3]

A *sectional* family is constructed by taking an arbitrary surface in space, drawing all the plane curves possible on the surface, and projecting these curves from any fixed center onto the (x, y)-plane. A triply infinite family of curves is thus produced, since there are ∞^3 plane sections of the surface. The projective nature of the construction is obvious. If the center of projection is taken to be the point at infinity on the z-axis, the projection becomes orthogonal, and the equation of the family of projected curves is found to be

$$(2)\quad y''' = \frac{f_{xxx} + 3f_{xxy}y' + 3f_{xyy}y'^2 + f_{yyy}y'^3}{f_{xx} + 2f_{xy}y' + f_{yy}y'^2}\, y'' + \frac{3(f_{xy} + f_{yy}y')}{f_{xx} + 2f_{xy}y' + f_{yy}y'^2}\, y''^2,$$

where $z = f(x, y)$ represents the surface.[4]

For instance, stereographic projection of the sphere produces the family of all circles.

Curvature trajectories may be described as an analogue for three-parameter families of what isogonal trajectories are for two-parameter families. Here we take as a basis an arbitrarily given *doubly* infinite family of curves. A new curve drawn at random would be tangent at each of its points to some curve of the base family. If now we require the new curve, say Γ, to be drawn in such a way that the ratio of its curvature at any point, to the curvature of the member of the base family which it touches at that point, shall be kept a fixed number c as we move along Γ, then we have a curvature trajectory of the base family. For any one value of c, ∞^2 trajectories can be drawn; hence ∞^3 in all. That the construction is projective follows from Mehmke's theorem, that the ratio of curvatures of curves tangent at a point is a projective invariant. The differential equation of a curvature family has the form

$$(3)\quad y''' = [(F_x + F_y y')/F]y'' + (F_{y'}/F)y''^2,$$

where $y'' = F(x, y, y')$ represents the arbitrary base family.[5]

For instance, if the base family is composed of all the unit circles, the curvature trajectories will be the family of all circles.

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[3] "De l'homographie en mecanique," *American Journal of Mathematics*, vol. 12 (1890), p. 103.

[4] Kasner, "Dynamical trajectories and the ∞^3 plane sections of a surface," *Proceedings of the National Academy of Sciences*, vol. 17 (1931), p. 370.

[5] Kasner, "Dynamical trajectories and curvature trajectories," *Bulletin of the American Mathematical Society*, vol. 40 (1934), p. 449. The ∞^2 straight lines of the plane appear as the one degenerate family under all three types. This family is excluded in the following.

We observe that for each of the three types just defined the differential equation has the following special form in y'':

(4) $$y''' = G(x, y, y')y'' + H(x, y, y')y''^2.$$

This special form is the analytic expression of the first of Kasner's set of six geometrical properties for dynamical families:

Property I. If to each of the ∞^1 curves having a given lineal element in common the osculating parabola is drawn at that element, the foci will lie on a circle through the point of the element.

The set of all families obeying Property I is once more a projective body. In fact equation (4) is invariant under the full projective group, correlations as well as collineations.[6]

3. We can now summarize the answers to the two sets of problems.

A1) The families of curves which are both dynamical and sectional are those derived from *cones* (of any cross section). They are the trajectories of a class of central fields of force of inverse square character.[7]

A2) The families of curves which are at once of the dynamical and curvature types are the trajectories of exactly all *central fields of force.*

A3) The families of curves which are of both the sectional and curvature types are those derived from *cones* and *quadric surfaces.*

B1) There are three cases in which all the trajectories in a field of force are conic sections:

a) Conics with a common polar pair (the point not on the line).
b) Conics tangent to two intersecting lines.
c) Conics tangent to a line at a point.

B2) There are four cases in which all the curves of a sectional family are conic sections. These are b) and c), and:

d) Conics through two fixed points.
e) Conics tangent twice to a fixed (proper) conic.

B3) There are five cases in which all the curves of a curvature family are conics. These are a), b), c), d), and e).

Equation (1) for dynamical trajectories, the definition of the sectional type, and the concept of curvature trajectories, as well as the solutions of problems A1, A2, and B2 are all due to Kasner. B1 is the well-known

[6] *Princeton Colloquium,* p. 77.

[7] The force equals $f(\theta)/r^2$. Without recognizing the connection with surfaces, Jacobi discussed these fields, showing that their trajectories are obtainable by quadratures. (*Werke,* IV, p. 282.)

Bertrand's problem, solved by Darboux and Halphen.[8] A3 and B3 are proved below, and we also discuss B2.

4. *Proof of* A3. We remark first that part of A3 is implied at once by the combination of A1 and A2, and that another part may be deduced from the circumstance that the family of all circles is an example of a sectional family and also of a curvature family.

If a family of curves belongs to both the sectional and curvature types, equations (2) and (3) must become identical.[9] We may therefore equate the coefficients of y'' and of y''^2. Integrating the latter equation, we find

$$(5) \qquad F(x, y, y') = (f_{xx} + 2f_{xy}y' + f_{yy}y'^2)^{3/2}\,\theta(x, y),$$

where $\theta(x, y)$ is arbitrary. We use this relation to eliminate F from the result of equating the coefficients of y''. After simplifying, we obtain a cubic polynomial in y' which must vanish identically. The four coefficients therefore vanish separately, and we have a system of four equations in f and θ, two of which are

$$(6) \qquad \bar{\theta}_x = -\tfrac{1}{2}f_{xxx}/f_{xx}, \qquad \bar{\theta}_y = -\tfrac{1}{2}f_{yyy}/f_{yy},$$

where $\bar{\theta} = \log\theta$. Using these two, we eliminate θ, and thus find that the following conditions must be satisfied by f:

$$(7_1) \qquad 2f_{xy}f_{yy}f_{xxx} - 3f_{xx}f_{yy}f_{xxy} + f^2{}_{xx}f_{yyy} = 0,$$

$$(7_2) \qquad f^2{}_{yy}f_{xxx} - 3f_{xx}f_{yy}f_{xyy} + 2f_{xx}f_{xy}f_{yyy} = 0,$$

$$(7_3) \qquad (f_{xxx}/f_{xx})_y = (f_{yyy}/f_{yy})_x.$$

The details of the elimination show that equations (7) are sufficient as well as necessary for the identity of (2) and (3). Hence the surfaces $z = f(x, y)$ which afford solutions of our problem are those defined by the system (7).

By differentiating (7_1) and (7_2) partially with respect to x and y, and combining, we find that they imply (7_3), unless

$$(8) \qquad f_{xx}f_{yy} - f^2{}_{xy} = 0;$$

that is, unless the surface is developable. On the other hand, (7_1) and (7_2) are themselves found to be consequences of (8). Thus there are two cases:

[8] *Comptes Rendus*, vol. 84 (1877), pp. 671, 731, 760, 936, 939. References in *Enc. der Math. Wiss.*, vol. IV 6, p. 498.

[9] A family derived by central projection from a surface S can always be considered as derived orthogonally from a surface S' projectively related to S.

We shall assume that the coefficients of y'' and y''^2 in equations (1), (2), (3), and (4) are analytic in x, y, and y'.

either the surface $z = f(x, y)$ is developable, and obeys (7_3) and (8); or it is not developable, and obeys (7_1) and (7_2).

In the first case we merely verify that (7_3) is equivalent under (8) to the condition $(f_{xxy}/f_{xy})_y = (f_{yyy}/f_{yy})_x$, which was obtained with (8) in the solution of problem A1. Hence the surfaces defined by (7_3) and (8) are those which appear in the answer to A1; namely, cones (and cylinders).

Equations (7_1) and (7_2) were derived by Hermite [10] and further discussed by Halphen.[11] Their solutions are quadrics and developables. Hence the solutions which they contribute to our problem are just quadrics. The conclusion is that the surfaces which generate curvature trajectories are cones and quadrics, as A3 states.

Proofs of A3 by direct integration of the differential equations have been given by M. Halperin and by the writer.

5. *Problem* B2. If every plane section of a surface X projects into a conic, every plane section of X is itself a conic. Then X must be a quadric surface. This may be shown, for instance, in the following way.

Let C_1 and C_2 be two conics on X, intersecting in two points J and K, and let P be a further point on X. A quadric Q can be passed through the configuration C_1C_2P. (We can take J, K, P, two other points M and N on the conics, and the tangent planes at J and K as determining elements.) Every plane through P which meets C_1 and C_2 in four points must cut out the same conic on Q as on X, since five points determine a conic. It follows that X must be Q.

The quadric may be proper or degenerate, and the center of projection may be on the surface or off. This accounts for the four projectively distinct cases b), c), d), and e); (it is easy to show synthetically that case e), for example, is obtained by projecting a proper quadric from an outside point). A unified description of these would be: conics twice tangent to a fixed curve of second order or second class, proper or degenerate; (the two most degenerate types may be said to appear coincidently in c)). The fixed locus, which may be imaginary, comes from the intersection of the quadric with the polar plane of the center of projection. It is the umbral curve, or boundary of the geometrical shadow, of the surface.

6. *Proof of* B3. In view of the previous results, a) to e) are already

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[10] *Cours d'Analyse*, vol. 1, p. 149. Hermite notes that (7_1) and (7_2) are the conditions for the denominator of the coefficient of y'' in (2) to divide the numerator.

[11] "Sur le contact des surfaces," *Bulletin de la Société Mathématique de France*, vol. 3 (1874), p. 28. Developables enter when the polynomial B (p. 33) is a perfect square.

known to be curvature trajectories, (for a), b) and c) in B1 do come from central fields of force). The effect of this proof is, then, to show that there are no other families of conics which are curvature trajectories.

We employ the differential equation of all conics:

$$(9) \qquad 9y''^2y^{v} - 45y''y'''y^{iv} + 40y'''^3 = 0.$$

If an equation of the form $y''' = Gy'' + Hy''^2$ represents a family of conics, it must render (9) an identity in x, y, y', y''. We therefore differentiate to find y^{iv} and y^{v}, and substitute in (9). The result is cubic in y''.[12] Setting the four coefficients equal to zero, we have:

$$(10_1) \qquad H_{y'y'} + 2HH_{y'} + \tfrac{4}{9}H^3 = 0,$$

$$(10_2) \qquad G_{y'y'} + GH_{y'} + \tfrac{1}{3}GH^2 + 2H_{xy'} + 2H_{yy'}y' + HH_x + HH_yy' + H_y = 0,$$

$$(10_3) \qquad H_{xx} + 2H_{xy}y' + H_{yy}y'^2 - G_xH - G_yHy' + \tfrac{1}{3}G^2H + 2G_{xy'} + 2G_{yy'}y' - GG_{y'} + G_y = 0,$$

$$(10_4) \qquad G_{xx} + 2G_{xy}y' + G_{yy}y'^2 - 2GG_x - 2GG_yy' + \tfrac{4}{9}G^3 = 0.$$

A cubic which vanishes for four values of the variable vanishes identically. Consequently:

If $4\infty^2$ out of the ∞^3 curves of a family having Property I are conics, the rest are conics also.[13]

Now in (10_1) only derivatives with respect to y' appear. Hence we can solve (10_1) as an ordinary differential equation, to find how H involves y'. The result is

$$(11_1) \qquad H = 3(\sigma + \tau y')/(\rho + 2\sigma y' + \tau y'^2),$$

where ρ, σ, τ are arbitrary functions of x and y. If $\rho\tau - \sigma^2 = 0$, the numerator divides the denominator, and (11_1) can be reduced to

$$(11_2) \qquad H = 3/(y' - \omega),$$

where $\omega(x, y)$ is arbitrary. (The case $H = 0$ is brought under (11_2) with $\omega = 0$ by interchanging x and y.)

[12] A factor y''^3 is dropped. (9) may be looked on as the condition for sextactic points. When (10_1) holds, one root of the cubic becomes infinite; then when (10_2) holds, two roots are infinite, etc. ($y'' = \infty$ represents a *point union.*)

[13] The ∞^2 straight lines, present in every Property I family, are not to be counted in the "$4\infty^2$". This can be strengthened to "$3\infty^2$" for the dynamical type or the sectional type, since both of these satisfy (10_1) identically. J. N. Hazzidakis gave the following theorem for the dynamical type: if for two directions of initial motion all the trajectories are conics, then the family is entirely composed of conics. (*Journal für Mathematik*, vol. 133 (1908), p. 68.)

These formulas have geometric meaning. The second one expresses Property II of Kasner's set for dynamical trajectories:

Property II. There exists for each point (x, y) of the plane a certain direction ω (that of the "*force*") such that the angle between this direction and the tangent to the focal circle (of Property I) corresponding to any element (x, y, y') at the given point, is bisected by that element.

It can be shown that formula (11_1) generalizes Property II in the following way:

There exist for each point (x, y) of the plane two directions ω_1 and ω_2 (the "*asymptotic*" directions) such that the angle between the tangent at (x, y) to the focal circle corresponding to any element (x, y, y'), and the harmonic conjugate of y' with respect to ω_1 and ω_2, is bisected by the element.[14]

When ω_1 and ω_2 coincide, this reduces to Property II. Evidently (11_1) holds for the sectional type (2). The integral curves of the directions ω_1, ω_2 are then the projections of the asymptotic lines on the surface (an *asymptotic net*), and the two harmonic conjugates come from conjugate directions on the surface.

The two forms (11) for H divide the problem naturally into what we may call a *sectional case* and a *dynamical case*.

Dynamical case. We substitute (11_2) into the second of equations (10), and obtain an ordinary differential equation for G as a function of y'. Solving, we find that

$$G = (\lambda y'^2 + \mu y' + \nu)/(y' - \omega), \tag{12}$$

where λ, μ, ν, ω are any functions of x and y satisfying

$$\lambda\omega^2 + \mu\omega + \nu - \tfrac{3}{2}(\omega_x + \omega\omega_y) = 0. \tag{13}$$

Again the equations allow of geometrical interpretation. (12) stands for the two equivalent dynamical properties, III and IV:

Property III. The locus of the centers of the ∞^1 circles corresponding to the lineal elements at a given point is a conic with that point as a focus.

Property IV. In each direction through a given point O there passes one trajectory which has contact of third order with its circle of curvature. The locus of the centers of the ∞^1 hyperosculating circles, obtained by varying the initial direction, is a conic passing through the given point in the direction of the force.

Property V. Of the curves which pass through a given point in the

[14] Kasner, *Bulletin of the American Mathematical Society*, vol. 14 (1908), p. 356; vol. 36 (1930), p. 51.

direction of the "force" at that point, there is one which has contact of the third order with its circle of curvature; the radius of curvature of this curve is three times the radius of curvature of the "line of force" (i. e., integral curve of the direction assigned to each point by II) passing through the given point.

The analytic statement of this last property is

$$\lambda\omega^2 + \mu\omega + \nu + \omega_x + \omega\omega_y = 0. \tag{14}$$

This differs only slightly from (13). We can see by following the argument in the reference that (13) changes Property V to this extent: the "three times" in V is replaced by "twice". It is clear that (14) holds in our problem only when $\omega_x + \omega\omega_y = 0$; that is, only when the "lines of force" defined by $y' = \omega(x, y)$ are straight lines.

At this point we recall that the families of conics we are finding are assumed to be curvature trajectories. A function $F(x, y, y')$ therefore exists such that

$$G = (F_x + F_y y')/F, \qquad H = F_{y'}/F, \tag{$15_{1,2}$}$$

as we see on comparing (3) and (4). From (15_2) and (11_2) we find that

$$F = (y' - \omega)^3\,\theta(x, y), \tag{16}$$

where $\theta(x, y)$ is arbitrary. Substituting this expression for F into (15_1), we have

$$G = [\bar{\theta}_y y'^2 + (\bar{\theta}_x - \omega\bar{\theta}_y - 3\omega_y)y' - (\omega\bar{\theta}_x + 3\omega_x)]/(y' - \omega), \tag{17}$$

where $\bar{\theta} = \log\theta$. We now equate coefficients in (17) and (12), and use (13). We find without difficulty that the existence of θ has the following two consequences:

$$\omega_x + \omega\omega_y = 0, \tag{18}$$

$$\lambda_x + [(\nu + 3\omega_x)/\omega]_y = 0. \tag{19}$$

The first of these, as we have said, means that Property V holds. It also means that the denominator of G in (12) divides the numerator; (13) shows this directly. Thus (12) reduces to

$$G = \lambda y' - \nu/\omega. \tag{20}$$

With (19) we may compare

$$\lambda_x + [(\nu + \omega_x)/\omega]_y = 0, \tag{21}$$

which is the analytic form of the final one of Kasner's set of properties of dynamical trajectories:

Property VI. When the point O is moved, the associated conic described

in Property IV changes in the following manner. Take any two fixed perpendicular directions for the x direction and the y direction; through O draw lines in these directions meeting the conic again at A and B, respectively. Also construct the normal at O meeting the conic again at N. At A draw a line in the y direction meeting this normal in some point A', and at B draw a line in the x direction meeting the normal in some point B'. The variation property referred to, takes the form

$$\frac{\partial}{\partial x}\frac{1}{AA'} + \frac{\partial}{\partial y}\frac{1}{BB'} + \frac{\omega\omega_{xy} - \omega_x\omega_y}{3\omega^2} = 0, \tag{22}$$

where AA' and BB' denote distances between points, and where ω denotes the slope of the lines of force relative to the chosen x direction. This is true for any pair of orthogonal directions, and therefore really expresses an intrinsic property of the system of curves. (See diagram in references.)

As before, we find that (19) is interpreted by simply deleting the 3 in the denominator of the last term of (22).[15] Thus (21) and (19) will both hold at once only when $\omega\omega_{xy} - \omega_x\omega_y = 0$; that is, using (18), only when $\omega_{yy} = 0$. But it is easy to show that $\omega_{yy} = 0$ and (18) imply that either $\omega(x, y) = y/x$ (essentially), or $\omega(x, y) = const.$ Hence to establish Property VI we must prove that the field of force is either central or parallel. To do this, however, we must use the third of equations (10).

We therefore set (20) and (11_2) into (10_3). We need only the two highest coefficients of the resulting identity in y'. From these, with the aid of (18) and (19), we deduce the following equations:

$$\lambda_x + \omega\lambda_y + \omega_y\lambda - \gamma_y - 3\omega_{yy} = 0, \tag{23}$$

$$\lambda(\gamma + 3\omega_y) - 3\gamma_y = 0, \tag{24}$$

$$3\gamma_x + 3\omega\gamma_y - \gamma^2 = 0, \tag{25}$$

where $\gamma = \lambda\omega - \nu/\omega$. ($\gamma$ represents G for the force direction.) Now (24) serves to eliminate λ from (23), assuming $\gamma + 3\omega_y \neq 0$. By means of (25) and (18) we then eliminate derivatives of γ and ω with respect to x. The result factors into

$$(\gamma + 3\omega_y)^2\,\omega_{yy} = 0. \tag{26}$$

If $\gamma + 3\omega_y = 0$, then differentiating, $\gamma_y + 3\omega_{yy} = 0$; but from (24), $\gamma_y = 0$. Hence in any case $\omega_{yy} = 0$. Property VI therefore does hold.

Since Properties I-VI form a characteristic set for dynamical trajectories, we have proved that in the dynamical case, curvature families of conics must

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[15] **It is curious that this change produces Property VI of *velocity curves*. Similarly, the change in V gave Property V of *catenaries*. (*Princeton Colloquium*, p. 94.)**

be dynamical trajectories; that is, they must be one of the three kinds a), b), and c) in B1. It follows from A2 (since a), b), c) come from central fields) that all three kinds are in fact curvature trajectories. Hence a), b), c) are the answers in the dynamical case.

We observe that the last point could be established independently of A2; for the process of applying the dynamical properties expressed in (4), (11_2), (12), (14), and (21) would closely parallel the work above, and would give exactly the same result. In other words, the above proof can be turned into a solution of the first part of Bertrand's problem, to show that fields of force whose trajectories are conics must be central or parallel.

Sectional case. As we do not have a set of properties for sectional families like those for dynamical families, we cannot give a geometrical interpretation for each step in this case. We outline the calculations briefly. A point of interest is that only (10_2) is used, whereas the dynamical case required (10_3) as well.

H is now supposed to be given by (11_1) with $\rho\tau - \sigma^2 \neq 0$. Applying ($15_2$), we determine the form of F in y'; this is found to be the same as in (5). Then from (15_1) we find that G has the same form in y' as the G of the sectional type (2). The expressions for H and G are now substituted into (10_2), and the resulting identity in y' yields a system of four equations from which all the unknowns can be calculated. In this way we obtain for F the following expression:

$$(27) \quad F = \left[\frac{(a_{33}y^2 - 2a_{23}y + a_{22}) - 2(a_{33}xy - a_{23}x - a_{13}y + a_{12})y' + (a_{33}x^2 - 2a_{13}x + a_{11})y'^2}{A_{11}x^2 + 2A_{12}xy + A_{22}y^2 + 2A_{13}x + 2A_{23}y + A_{33}} \right]^{3/2},$$

where $a_{11}, \cdots, a_{33}$ are arbitrary constants except that $|a_{ij}|$, (where $a_{ji} = a_{ij}$), is of rank at least 2, and where $A_{11}, \cdots, A_{33}$ are cofactors in $|a_{ij}|$.

The final step consists in forming the F for an arbitrary proper quadric, (say by means of the formula implied in the proof of A3), and identifying it with (27). Two cases are necessary, depending on whether or not the quadric contains the point at infinity on the z-axis (the center of projection). The identifying requires a theorem on minors in an adjoint determinant; (Bôcher, *Algebra*, p. 31).

The families of conics obtained in this case are therefore d) and e) of B2. The proof of B3 is thus complete.

7. Part of the last derivation applies to a more general problem: to determine all families of conics having Property I. This includes B1, B2,

and B3. The answers here can be converted by duality, since Property I is invariant under correlations.[16]

We give examples to show that the solutions to the proposed problem will add to the set a), b), c), d), e). The following triply infinite families of conics have Property I, but are neither dynamical nor sectional:

f) Let points A, B, C be the vertices of a triangle. A conic through A will cut sides AB and AC again in points M and N. The family consists of the ∞^3 conics through A for which the tangents at M and N meet on BC.

g) Let the pencil of lines through a fixed point A be in projective correspondence with the points of a fixed line l through A, in such a way that l corresponds to A. A conic through A will be tangent at A to a line u of the pencil, and will cut l again in a point X. The family consists of the ∞^3 conics through A for which u and X correspond.

An alternative form of Property I states that the ∞^1 osculating parabolas at a given lineal element have concurrent directrices.[17] Family f) may be considered an obvious construction from this form of Property I; for if we dualize f) and specialize the triangle properly, we obtain the family of ∞^3 parabolas whose directrices pass through a given point. Family g) is a limiting case of f). Both f) and g) arise in the sectional case of the problem. They are the only new solutions for which one set of "asymptotic curves" is a pencil of straight lines.

In the dynamical case, it can be shown that there are no new solutions, under the assumption that the "lines of force" are a family of straight lines. An equivalent statement is this: Properties I-V suffice to define a), b) and c) as the only families of conics which are dynamical trajectories. That is, in solving Bertrand's problem it is not necessary to impose all the six geometrical properties of dynamical trajectories, for the sixth property becomes a consequence of the other five.

An example of a new family in the dynamical case is given by the ∞^3 parabolas with directrices through a fixed point. The "lines of force" are the ∞^1 circles concentric about the point. This family has Properties I, II, III, IV, and also VI, but not V. It is therefore as nearly dynamical as any non-dynamical family of conics can be, in the sense that it has a maximum number of the six properties without having all of them. It might be termed the next thing to a Bertrand family.

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[16] If we apply Legendre's transformation (polarity with respect to the parabola $x^2-2y=0$) to equations (10), and interchange G with $-H$, we obtain the same equations in reverse order. This expresses the fact that the dual of a Property I family of conics is such a family. (See *Princeton Colloquium*, p. 78.)

[17] *Princeton Colloquium*, p. 12.

SOME INTERPRETATIONS OF ABSTRACT LINEAR DEPENDENCE IN TERMS OF PROJECTIVE GEOMETRY.[1]

By Saunders MacLane.

1. *Introduction.* The abstract theory of linear dependence, in the form recently developed by Whitney,[2] is closely related to the study of projective configurations. For any matroid (that is, any finite system of elements for which a suitably restricted notion of "linear dependence" is given) can be interpreted as a schematic geometric figure. Such a schematic figure, like a schematic configuration, is composed of a number of points, lines, planes, etc., with certain combinatorially defined incidences. The problem of representing a matroid by a matrix then becomes simply the problem of realizing a schematic figure by some geometric figure—and the impossibility of always finding such a representation turns out to be a simple consequence of Pascal's theorem! Even when such representation is possible, it depends essentially upon the field from which the elements of the representing matrix are taken. However, only algebraic fields need be used, and hence arises a connection between certain matroids and the algebraic fields in which they can be best represented.

Matroids will be defined by axioms on "rank," as in Whitney's paper. Without loss of generality we can also assume the following two axioms:

R_4: The rank of a single element is always 1.

R_5: The rank of a pair of elements is always 2.

For example, an element e which does not satisfy R_4 may be dropped from or added to a matroid M without otherwise altering the structure of M. These two axioms are equivalent to the following conditions on "bases":

B_3: Every element belongs to at least one base.

B_4: There is no pair of elements (e_1, e_2) such that every base containing e_1 remains a base when e_1 is replaced by e_2.

These conditions on M are in turn equivalent to the following conditions on the dual matroid M^*:

C^*_3: Every element is omitted from at least one circuit complement.

[1] Presented to the American Mathematical Society, December 28, 1934.

[2] H. Whitney, "On the abstract properties of linear dependence," *American Journal of Mathematics*, vol. 57 (1935), pp. 509-533.

C^*_4: For every pair of elements (e_1, e_2) there is a circuit complement containing e_1 but not e_2.

2. *Schematic geometric figures.* A rectilinear plane figure consists of a number of points and of all the lines joining these points in pairs. The combinatorial structure of such a figure can be specified by giving for each line L the set of all those points of the figure which lie on L. These sets satisfy the following axioms:

F_1: Any pair of points belongs to one and only one line.
F_2: Every line contains at least two points.
F_3: No line contains all the points.
F_4: There are at least two points.

A system consisting of a finite number of "points" and certain sets of these points, called "lines," and satisfying these axioms will be called a *schematic plane figure*; it may or may not correspond to some actual figure. In the same way a *schematic space figure* would involve "points," "lines," and "planes," satisfying F_1 to F_4 and the following axioms:

S_1: Every triple of points belonging to no line belongs to one and only one plane.
S_2: Every plane contains three points not on a line.
S_3: No plane contains all the points.
S_4: If a plane contains two points of a line, it contains all the points of that line.

The definition of a schematic n-dimensional figure is similar; it involves "points" and "k-dimensional planes" for $k = 1, \cdots, n-1$.

The equivalence of schematic figures and matroids may be formulated as follows:

THEOREM 1. *Every schematic n-dimensional figure is a matroid of rank $n+1$ if the rank of a set of points A is defined as the smallest r such that all the points of A are contained in some $(r-1)$-plane. Conversely, every matroid of rank $n+1$ becomes a schematic n-dimensional figure if the k-planes are taken as maximal sets of elements of rank $k+1$. This translation sets up a one-one correspondence between matroids and schematic figures.*[3]

From this theorem it follows that a schematic n-dimensional figure is

[3] The conditions R_4 and R_5 of § 1 are necessary here to exclude the geometrically meaningless cases of a point of dimension -1 or of two coincident points.

determined once the corresponding matroid M or its dual M^* is given. By Whitney's results, this dual M^* is completely determined by its circuit complements. The circuit complements in M^* correspond to maximal sets of rank $r(M) - 1$ in M. Hence

THEOREM 2. *A schematic n-dimensional figure is completely determined if its $(n-1)$-planes are given. If a set of "points" and certain subsets of this set are given, these subsets will be the $(n-1)$-planes of some figure if and only if they are the circuit complements of a matroid M^*; that is, if and only if these subsets satisfy the above axioms C^*_3 and C^*_4, while their complements satisfy Whitney's axioms C_1 and C_2 for circuits.*

These results also show that matroids form a direct generalization of schematic configurations. A *schematic plane configuration* [4] $p_\gamma g_\pi$ consists of p "points" and g "lines," with each point on γ lines and each line on π points. Such a configuration becomes a schematic figure in the above sense if those pairs of points not already joined by lines are joined by new "diagonal" lines. Similar transformations are possible for space configurations.

3. *Matrix representations of matroids.* The columns of a matrix stand in relations of rank and thus form a matroid. The question whether every matroid can be represented in this way by a matrix is clarified by the equivalence of matroids and schematic figures. Thus Whitney has constructed a matroid of rank 3 which cannot be represented as a matrix. This matroid has 9 elements $1, 2, \cdots, 9$ and the following 20 circuit complements:

$$712,\ 814,\ 923,\ 734,\ 836,\ 945,\ 756,\ 825;$$
$$16,\ 19,\ 69,\ 13,\ 15,\ 24,\ 26,\ 35,\ 46,\ 78,\ 79,\ 89.$$

Any attempt to represent this matroid yields a figure in which the lines 16, 19, and 69 coalesce into one line 169. A geometric representation reveals at once that this is simply Pascal's theorem for the hexagon 723845 inscribed in the degenerate conic composed of the two lines 743 and 825. The points 1, 6, and 9 are the intersections of opposite sides of the hexagon. In exactly the same way the theorem of Desargues may be used to construct a matroid with ten elements which has no matrix representation. Furthermore, the matroid arising from Pascal's theorem can be generalized to the case of $2m + 3$ elements, which we denote by $1, 2, \cdots, 2m, \alpha, \beta, \gamma$. The circuit complements are:

[4] F. Levi, *Geometrische Konfigurationen*, p. 4.

$$12\alpha,\ 34\alpha, \cdots, (2m-3, 2m-2, \alpha), (2m-1, 2m, \alpha),$$
$$14\beta, 36\beta, \cdots, (2m-3, 2m, \beta), (2m-1, 2, \beta),$$
$$23\gamma, 45\gamma, \cdots, (2m-2, 2m-1, \gamma).$$

together with all the pairs of elements not included in one of these triples. No matrix representation is possible, for any attempt to construct one yields a matroid with the additional circuit complement $(2m, 1, \gamma)$.

These matroids fail to be matrices because of the presence of too few circuit complements. Failure is also possible for the opposite reason. Thus the plane figures (matroids) formed by finite projective geometries [5] can be represented only by matrices of elements from a finite field. Another important special case of the matrix representation of matroids is the problem of constructing a geometric realization for schematic plane configurations. Here it is well-known that a configuration (p_γ, g_π) cannot in general be realized if [6]

$$2(p+g) - p\gamma - 8 < 0.$$

The use of geometric figures also simplifies the investigation of the conditions for the representability of individual matroids. Thus for a matroid M of rank 3 we need only find three homogeneous coördinates for each element (point) of the matroid, such that when three points lie on a line (i. e., are contained in a circuit complement of the dual matroid), then the determinant of the corresponding coördinates is zero, and conversely. This application of the usual theorems of analytic geometry can replace Whitney's Theorem 32.

4. *Representation in finite algebraic fields.* The configuration of eight elements which can be represented in the complex but not in the real plane [7] suggests that the representability of a matroid depends essentially on the field from which the elements of the representing matrix are taken. Another similar example can be constructed for the field $R(2^{1/2})$, where R is the field of rational numbers. We need only take a point with coördinates $(1, 2^{1/2}, 0)$ and carry out the constructions in the von Staudt algebra of throws corresponding to

$$(2^{1/2})(2^{1/2}) = 1 + 1.$$

The resulting figure (matroid) consists of 11 points, $1, 2, \cdots, 9, 0, \alpha$, the following sets of points being collinear:

$$1279\alpha,\ 2356,\ 1380,\ 248,\ 347,\ 578,\ 549,\ 690,\ 50\alpha,\ 68\alpha.$$

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[5] Veblen and Young, *Projective Geometry*, vol. I, p. 3 and p. 201.

[6] E. Steinitz, *Encyklopädie der mathematischen Wiss.*, III A B 5a, p. 485.

[7] F. Levi, *loc. cit.*, pp. 98-102.

Any attempt to represent this matroid by a matrix leads to a matrix whose elements generate a field containing $R(2^{1/2})$. This matroid is thus a sort of geometric analog of the irreducible equation for $2^{1/2}$. Generalization yields:

THEOREM 3. *Let K be a finite algebraic field over the field of rational numbers. Then there exists a matroid M of rank 3 which can be represented by a matrix with elements in K, while any other representation of M by a matrix with elements in a number-field K_1 requires $K_1 \supset K$.*

Such finite fields are sufficient for the representation of all representable matroids, in the following sense.

THEOREM 4. *Let the matroid M be representable by a matrix of complex numbers. Then M can also be represented by a matrix with elements from an algebraic field of finite degree.*

For let the matroid M have rank n and consist of p points, and let these points be assigned the indeterminate homogeneous coördinates a_{ij}, for $i = 1, \cdots, n$; $j = 1, \cdots, p$. Each circuit complement of the dual of M requires the vanishing of a number of determinants of the a_{ij}, and thus corresponds to a number of algebraic equations for these quantities. The set of all values of the a_{ij} giving at least the required circuit complements thus constitutes an algebraic manifold N_1 in the np-dimensional space of all coördinates a_{ij}. The set of those coördinates yielding additional undesired circuit complements forms another manifold N_2. Since the original matrix was representable, there exists a point of $N_1 - N_2$. The parametric representation of irreducible algebraic manifolds [8] thus makes possible the construction of a point with algebraic coördinates in $N_1 - N_2$.

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[8] B. L. van der Waerden, *Moderne Algebra*, vol. 2, p. 51 ff.

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